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Super self-duality for Yang-Mills fields in dimensions greater than four*

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Abstract: Self-duality equations for Yang-Mills fields in d-dimensional euclidean spaces consist of linear algebraic relations amongst the components of the curvature tensor which imply the Yang-Mills equations. For the extension to superspace gauge fields, the *super self-duality* equations are investigated, namely, systems of linear algebraic relations on the components of the supercurvature, which imply the self-duality equations on the even part of superspace. A group theory based algorithm for finding such systems is developed. Representative examples in various dimensions are provided, including the Spin(7) and G_2 invariant systems in $d = 8$ and 7, respectively.

Keywords: Field Theories in Higher Dimensions, Duality in Gauge Field Theories, Supersymmetry and Duality.

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Contents

1. Introduction

1.1 Generalised self-duality

Supersymmetric instanton-like solutions and BPS states in dimensions greater than four have recently drawn increased attention. For pure Yang-Mills theories in d-dimensional euclidean space, these are solutions of generalised self-duality equations, which were introduced in [1],

$$\frac{1}{2} T_{MNPQ} F_{PQ} = \lambda F_{MN} , \tag{1.1}$$

where the constant λ is non zero. The Yang-Mills curvature tensor F_{MN} takes values in the Lie algebra of some gauge group and the vector indices $M, N, \ldots$ run from 1 to d. Here T_{MNPQ} is a fourth rank completely antisymmetric SO(d) tensor. It has some stability group H $\subset$ SO(d), under which the $d(d-1)/2$-dimensional adjoint representation A of SO(d), corresponding to the space of skewsymmetric tensors $\Lambda_{MN} = -\Lambda_{NM}$, decomposes into a set $\rho_H(A)$ of irreducible representations $\underline{a}$,

$$A = \bigoplus_{\underline{a}\in\rho_H(A)} \underline{a}\,. \tag{1.2}$$

Consider the eigenvalue equation [1]

$$\frac{1}{2}T_{MNPQ}\Lambda_{PQ} = \lambda\Lambda_{MN}\,. \tag{1.3}$$

Each eigenvalue λ of T is associated to a subset of H-representations $\rho_H(\lambda) \subset \rho_H(A)$. The corresponding eigensolutions $(\Lambda^{\underline{a}})^l_{MN}$ may be labeled in terms of the associated irreducible H-representations $\underline{a} \in \rho_H(\lambda)$, with components labeled by the index l whose range is $\dim(\underline{a})$, the dimension of $\underline{a}$. The idea of [1] was to apply the eigenvalue eq. (1.3) to a Yang-Mills curvature tensor F_{MN} in d-dimensional euclidean space, obtaining the self-duality conditions (1.1). In general, the decomposition of the curvature into H-irreducible pieces reads

$$F_{MN} = \sum_{\underline{a}\in\rho_H(A)} F_{MN}(\underline{a})\,; \qquad F_{MN}(\underline{a}) := (\Lambda^{\underline{a}})^l_{MN}\,F^{(\underline{a})}_l \tag{1.4}$$

and the application of (1.1) for a specified eigenvalue λ means that the components of F live entirely in the corresponding T-eigenspace, i.e.

$$F_{MN} = \sum_{\underline{a}\in\rho_H(\lambda)} F_{MN}(\underline{a})\,. \tag{1.5}$$

This is equivalent to the statement that other parts of F in the decomposition (1.4) are set to zero,

$$F_{MN}(\underline{a}) = 0\,, \qquad \text{for all } \underline{a} \in \rho_H(\lambda)^{\complement}\,, \tag{1.6}$$

where the complementary set of representations $\rho_H(\lambda)^{\complement} = \rho_H(A)\backslash\rho_H(\lambda)$. The requirement that F lives in an eigenspace of T with a given non-zero eigenvalue λ is sufficient to guarantee that the Yang-Mills equations

$$D_M F_{MN} = 0 \tag{1.7}$$

are satisfied in virtue of the Bianchi identities

$$D_M F_{NP} + D_N F_{PM} + D_P F_{MN} \equiv 0\,. \tag{1.8}$$

Thus, the standard (4-dimensional) notion of self-duality (with $T_{MNPQ} = \epsilon_{MNPQ}$ and $\lambda = \pm 1$) was generalised to higher dimensions in [1]. There, examples in dimensions up to eight were also given.

1.2 Super self-duality

The purpose of this paper is to describe the generalisation of the above approach to superspace. In order to include spinors, the group SO(d) above is replaced by the universal covering group Spin(d). We will again denote by H the relevant stability subgroup. For super-Yang-Mills theories formulated in superspace, the vector potential is determined by fermionic spinor potentials. We address the question of determining systems of algebraic constraints on the "lower level" spinor-spinor and spinor-vector curvature components, which guarantee that the vector-vector curvature components automatically satisfy (1.1). These yield systems of lower order equations which automatically imply (1.1). We will primarily be concerned with identifying the simplest algebraic sets of such curvature constraints. We note that the question may be posed (and answered) in any dimension, independent of the existence of an underlying (fully supercovariant) Yang-Mills lagrangian field theory, which only exists in dimensions $d = 3, 4, 6, 10$.

Consider a superspace with d-dimensional euclidean even part with tangent space spanned by tangent vectors $\nabla^{(V)}_M$, $M = 1, \ldots, d$, which are components of the bosonic translation generator transforming according to the standard d-dimensional vector representation. The odd part of the tangent space is spanned by the components of N copies of fermionic translation operators, $\nabla^{(S)i}_B$, $i = 1, \ldots, N$, $B = 1, \ldots, \dim S$. These transform according to spinor representations S belonging to the set Σ of irreducible fundamental spinor representations of $\mathrm{Spin}(d)$. The supertranslation operators $\nabla^{(S)i}_B, \nabla^{(V)}_M$ generalise the derivatives.

We assume that the vector module V occurs in the product of spinor modules S and S', which may transform according to either distinct or identical irreducible fundamental $\mathrm{Spin}(d)$ representations, depending on the dimension d. (We discuss the minimal possibilities in appendix A.3). Accordingly, we demand

$$\left\{\nabla^{(S)i}_B, \nabla^{(S')j}_C\right\} = a^{ij} C\left(S, B, S', C; V, M\right) \nabla^{(V)}_M\,, \tag{1.9}$$

where a^{ij} is a numerical matrix which can be put in some canonical form appropriate to the dimension (and hence to its symmetry) and $C(S, B, S', C; V, M)$ is the Clebsch-Gordon coefficient extracting the vector V from the spinor representations $S, S' \in \Sigma$. In terms of realisations of Clifford algebras, the latter is simply the familiar gamma matrix $(\Gamma^M)_{BC}$. Since we do not require any particular properties of the gamma matrices, we shall use an abstract "Clebsch-Gordon" notation. For simplicity, we shall assume that apart from (1.9), the supertranslation operators mutually supercommute (i.e. commute or anticommute in accordance with their statistics). However, our considerations are independent of the possiblity that supercommutation relations among the spinors yield additional charges,which are central with respect to the supertranslation operators.

Gauge covariant derivatives are defined as $D^{(X)} = \nabla^{(X)} + A^{(X)}$, $X = V, S$, $(S \in \Sigma)$, where the gauge potentials $A^{(X)}$ take values in the Lie algebra of the gauge group and have the same $\mathrm{Spin}(d)$ behaviour and statistics as the corresponding derivative operators. The supercommutators of these operators yield the covariant spinor-spinor, spinor-vector and vector-vector supercurvature components, which take values in the Lie algebra of the gauge group. Generically, for $S, S' \in \Sigma$, we have,

$$\left\{D^{(S)i}_B, D^{(S')j}_C\right\} = a^{ij} C\left(S, B, S', C; V, M\right) D^{(V)}_M + \sum_{W \in \{S \otimes S'\}} C\left(S, B, S', C; W, L\right) F^{(W)ij}_L\,,$$

$$\left[D^{(S)i}_B, D^{(V)}_M\right] = \sum_{T \in \{S \otimes V\}} C(S, B, V, M; T, D)\ F^{(T)i}_D\,, \qquad \left[D^{(V)}_M, D^{(V)}_N\right] = F^{(A)}_{MN}\,, \tag{1.10}$$

where $\{X \otimes Y\}$ denotes the set of irreducible Spin(d) representation spaces Z appearing in the decomposition of the tensor product $X \otimes Y$ and having the appropriate symmetries. We have denoted by $C(X,Q,Y,R;Z,P)$ the Spin(d) Clebsch-Gordon coefficients corresponding to the projection to irreducible representation Z, with states labeled by the index P, in the tensor product of irreducible representations X and Y, labeled by indices Q and R, respectively. The curvature $F^{(A)}_{MN}$ is antisymmetric in M,N, transforming according to the adjoint representation A of Spin(d). The lower order curvatures $F^{(W)ij}_L$ are bosonic (with components indexed by L) and $F^{(T)i}_D$ are fermionic (with components indexed by D). Summation over repeated indices $M,\ldots$ labeling the states of representations is understood. It is obvious, in view of their construction, that the complete set of unconstrained covariant derivatives automatically satisfy the super Jacobi identities, which are merely associativity properties of the ∇'s and of the potentials. These provide super Bianchi identities for the curvature components. We note that for the $W=V$ term on the right-hand side of (1.10) there is an ambiguity: Any gauge covariant addition $\alpha^{(V)}_M$ to the vector potential $A^{(V)}_M$ can be compensated in the curvature by the shift $F^{(V)ij}_M \to F^{(V)ij}_M - a^{ij}\alpha^{(V)}_M$. Conversely, if there is only one $F^{(V)}_M$, it can be put to zero by absorbing it in $A^{(V)}_M$.

The super Jacobi identity between $D^{(S)i}_B, D^{(S')j}_C$ and $D^{(V)}_M$ yields the relationship

$$\begin{aligned} a^{ij} C\left(S,B,S',C;V,P\right) F^{(A)}_{PM} = & \sum_{W\in\{S\otimes S'\}} C\left(S,B,S',C;W,L\right)\left[D^{(V)}_M, F^{(W)ij}_L\right] + \\ & + \sum_{T\in\{S\otimes V\}} C(S,B,V,M;T,D)\left\{D^{(S')j}_C, F^{(T)i}_D\right\} + \\ & + \sum_{T'\in\{S'\otimes V\}} C\left(S',C,V,M;T',D\right)\left\{D^{(S)i}_B, F^{(T')j}_D\right\}. \end{aligned} \tag{1.11}$$

Clearly, the lower level curvature components $F^{(W)}, F^{(T)}$ determine the standard (vector-vector) curvature tensor $F^{(A)}_{NM}$. In fact there is a hierarchy of implications, since the further super Jacobi identities among three spinorial derivatives $D^{(S)}_B$ yields $F^{(T)}$ in terms of $F^{(W)}$. We define *super self-duality* as any system of algebraic conditions on the curvature components $F^{(W)ij}_L, F^{(T)i}_D$, which implies that $F^{(A)}_{NM}$ automatically satisfies (1.1) for a particular non-zero eigenvalue λ. The aim of this paper is to investigate such systems of sufficient conditions.

As we have seen, the self-duality condition (1.1) corresponds to the projection of the adjoint representation to the space of a subset of representations $\rho_H(\lambda) \subset \rho_H(A)$ covariant under a subgroup $H \subset$ Spin(d). Now under this subgroup H, every irreducible representation Z of Spin(d) decomposes into a direct sum of irreducible representations of H which we denote by $\underline{z}$:

$$Z = \bigoplus_{\underline{z}\in\rho_H(Z)} \underline{z}\,. \tag{1.12}$$

In particular, all potentials, covariant derivatives and curvatures decompose into irreducible components under H. We denote the descendant of a field $F^{(Z)}$, which transforms according to representation $\underline{z} \in \rho_H(Z)$, as $F^{(Z)}(\underline{z})$. In order to find the super self-duality conditions, we determine the parts of $\rho_H(W), \rho_H(T)$ which contribute to $\rho_H(\lambda)^{\complement}$ and those which contribute to $\rho_H(\lambda)$. Setting the parts of $F_L^{(W)ij}, F_D^{(T)i}$ contributing to $\rho_H(\lambda)^{\complement}$ to zero yields the required super self-duality equations. The parts contributing to $\rho_H(\lambda)$, which do not also contribute to $\rho_H(\lambda)^{\complement}$, act as sources for the components of $F_{NM}^{(A)}$ transforming according to $\rho_H(\lambda)$, i.e. satisfying (1.1). In this paper, we do not pursue the question of the relationship of our super self-duality equations with integrability conditions for "Killing spinors" (parameters of supersymmetry transformations) allowed by solutions of (1.1). We remark that if S is a spinor representation of Spin(d), its decomposition $\rho_H(S)$ need not always include spinor representations $\underline{s}$ of H.

We shall see that in order to determine the algebraic conditions on $F_L^{(W)ij}$ which determine $F_D^{(T)i}$, which in turn determine $F_{NM}^{(A)}$, we need to analyse super Jacobi identities of two types:

- *Level one identities* arise from the associativity of two spinorial covariant derivatives and the vectorial one (like (1.11))
- *Level two identities* arise from the associativity of three spinorial covariant derivatives.

We shall call the corresponding sets of supercurvature constraints, the level one and level two super self-duality equations. The level two equations imply the level one equations, which in turn imply the *level zero* self-duality eqs. (1.1) for the superfield $F_{MN}^{(A)}$. The latter clearly implying the superfield Yang-Mills equations $D_M^{(V)} F_{MN}^{(A)} = 0$ in virtue of the *level zero* Jacobi identities (1.8) arising from the associativity of three vectorial covariant derivatives. The precise form of the level one and level two super Jacobi identities depend on which irreducible part of $\Sigma \otimes \Sigma$ contains the vector module V. The tensor products for spinor representations of Spin(d) in appendix A shows that there are essentially three different cases, depending on whether $V = R(\pi_1) = (10 \cdots 0)$ is a submodule of the tensor product of two inequivalent spinor modules, or in the antisymmetric or symmetric part of the tensor product of a spinor module S with itself. Respectively, for $d \geq 4$, we have

- $d = 4 \pmod 4$, $V \subset S^+ \otimes S^-$
- $d = 5, 6, 7 \pmod 8$, $V \subset S \wedge S$
- $d = 9, 10, 11 \pmod 8$, $V \subset S \vee S$.

In the following sections, we examine the relevant super Jacobi identities in these three cases. Using group theory based algorithms, we develop a scheme for finding possible sets of super self-duality equations. Our main results consist of explicit examples of such systems of equations for specific choices of dimension d and subgroup H$\subset$ Spin(d). For $d = 4$ we shall consider the case of general extension N of the superspace. However, for higher d, we shall restrict our attention to the simplest available possibilities: $N = 2$ for the cases ($d = 5, 6, 7 \pmod 8$) in which the vector module V appears as a subspace of the antisymmetrised square of a spinor module; and $N = 1$ for all other cases. It is not difficult to extend our discussion to higher N.

2. The $d = 4$ case, $\mathbf{H} = \mathrm{Spin}(4)$

In this section we discuss the familiar $d = 4$ case in full detail, for general N-extension. This case illustrates well the pattern we have in mind: the standard self-duality equations for the vector potentials are implied by sets of sufficient conditions which are equations for the odd superpotentials. Consider 4-dimensional superspace with N-multiples of the two types of spinor representations. We use Dynkin indices (p,q) to denote $\mathrm{SU}(2) \times \mathrm{SU}(2)$ representations of dimension $(p+1)(q+1)$. We drop the representation labels V, S, etc. on the covariant derivatives and fields, since these are redundant in 2-spinor index notation. Let $\nabla^i_\alpha, \nabla^i_{\dot\alpha}$ $(i = 1, \ldots, N)$ be the $2N$ fermionic spinor operators, transforming, respectively, as the 2-dimensional (1,0) and (0,1) representations. Let $\nabla_{\alpha\dot\beta}$ be the lone bosonic vector operator, transforming as the (1,1) representation. We assume that all these operators commute or anticommute in agreement with their statistics except for

$$\left\{\nabla^i_\alpha, \nabla^j_{\dot\alpha}\right\} = \delta^{ij}\nabla_{\alpha\dot\alpha}\,. \tag{2.1}$$

On the right-hand side, by independent linear transformations of ∇^i_α and $\nabla^j_{\dot\alpha}$, more general coefficients a^{ij} can seen to be equivalent to δ^{ij} provided $\det a \neq 0$. These operators obviously form an associative algebra as all the super-Jacobi identities are trivially satisfied. Here, the Clebsch-Gordon coefficients and tensor products are easy to evaluate explicitly by 2-spinor index manipulations.

The supercommutators of the $2N + 1$ super covariant derivatives $\{D_{\alpha\dot\alpha}\,,\, D^i_\alpha, D^i_{\dot\alpha}\ ;\, i = 1, \ldots, N\}$ involve $2N+1$ potentials with the same indices and the same fermionic or bosonic behaviour as the corresponding ∇. They define the covariant spinor-spinor, spinor-vector and vector-vector curvature components. Thus, we have,

$$\begin{aligned}
\left\{D^i_\alpha, D^j_{\dot\alpha}\right\} &= \delta^{ij} D_{\alpha\dot\alpha} + F^{ij}_{\alpha\dot\alpha}\,, && W = S^+ \otimes S^- = (1,1)\,,\\
\left\{D^i_\alpha, D^j_\beta\right\} &= F^{ij}_{\alpha\beta} + \epsilon_{\alpha\beta}\, F^{ij}\,, && U^+ \in \left\{S^+ \otimes S^+\right\} = \{(2,0), (0,0)\}\,,\\
\left\{D^i_{\dot\alpha}, D^j_{\dot\beta}\right\} &= F^{ij}_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta}\, \overline{F}^{ij}\,, && U^- \in \left\{S^- \otimes S^-\right\} = \{(0,2), (0,0)\}\,,\\
\left[D^i_{\dot\alpha}, D_{\beta\dot\beta}\right] &= F^i_{\beta\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta}\, F^i_\beta\,, && T^+ \in \left\{S^- \otimes V\right\} = \{(1,2), (1,0)\}\,,\\
\left[D^i_\alpha, D_{\beta\dot\beta}\right] &= F^i_{\alpha\beta\dot\beta} + \epsilon_{\alpha\beta}\, F^i_{\dot\beta}\,, && T^- \in \left\{S^+ \otimes V\right\} = \{(2,1), (0,1)\}\,,\\
\left[D_{\alpha\dot\alpha}, D_{\beta\dot\beta}\right] &= \epsilon_{\alpha\beta} F_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta} F_{\alpha\beta}\,, && A \in \{V \wedge V\} = \{(0,2), (2,0)\}\,.
\end{aligned} \tag{2.2}$$

These eleven curvature tensors have the following symmetry in their indices

$$\begin{aligned}
F^{ij} &= -F^{ji}\,, \qquad \overline{F}^{ij} = -\overline{F}^{ji}\,,\\
F_{\alpha\beta} &= F_{\beta\alpha}\,, \qquad F_{\dot\alpha\dot\beta} = F_{\dot\beta\dot\alpha}\,,\\
F^{ij}_{\alpha\beta} &= F^{ij}_{\beta\alpha} = F^{ji}_{\alpha\beta}\,, \qquad F^{ij}_{\dot\alpha\dot\beta} = F^{ij}_{\dot\beta\dot\alpha} = F^{ji}_{\dot\alpha\dot\beta}\,,\\
F^i_{\alpha\beta\dot\beta} &= F^i_{\beta\alpha\dot\beta}\,, \qquad F^i_{\beta\dot\alpha\dot\beta} = F^i_{\beta\dot\beta\dot\alpha}
\end{aligned} \tag{2.3}$$

and hence behave irreducibly under Spin(4). The supercurvature components extracted from (2.2) are:

$$
\begin{aligned}
F^{ij}_{\alpha\dot\alpha} &= \nabla^i_\alpha A^j_{\dot\alpha} + \nabla^j_{\dot\alpha} A^i_\alpha + \left\{A^i_\alpha, A^j_{\dot\alpha}\right\} - \delta^{ij} A_{\alpha\dot\alpha}\,,\\
F^{ij}_{\alpha\beta} &= \frac{1}{2}\left(\nabla^i_\alpha A^j_\beta + \nabla^j_\beta A^i_\alpha + \left\{A^i_\alpha, A^j_\beta\right\} + \nabla^i_\beta A^j_\alpha + \nabla^j_\alpha A^i_\beta + \left\{A^i_\beta, A^j_\alpha\right\}\right),\\
F^{ij} &= \frac{1}{2}\epsilon^{\beta\alpha}\left(\nabla^i_\alpha A^j_\beta + \nabla^j_\beta A^i_\alpha + \left\{A^i_\alpha, A^j_\beta\right\}\right),\\
F^{ij}_{\dot\alpha\dot\beta} &= \frac{1}{2}\left(\nabla^i_{\dot\alpha} A^j_{\dot\beta} + \nabla^j_{\dot\beta} A^i_{\dot\alpha} + \left\{A^i_{\dot\alpha}, A^j_{\dot\beta}\right\} + \nabla^i_{\dot\beta} A^j_{\dot\alpha} + \nabla^j_{\dot\alpha} A^i_{\dot\beta} + \left\{A^i_{\dot\beta}, A^j_{\dot\alpha}\right\}\right),\\
\overline{F}^{ij} &= \frac{1}{2}\epsilon^{\dot\beta\dot\alpha}\left(\nabla^i_{\dot\alpha} A^j_{\dot\beta} + \nabla^j_{\dot\beta} A^i_{\dot\alpha} + \left\{A^i_{\dot\alpha}, A^j_{\dot\beta}\right\}\right),\\
F^i_{\alpha\beta\dot\beta} &= \frac{1}{2}\left(\nabla^i_\alpha A_{\beta\dot\beta} - \nabla_{\beta\dot\beta} A^i_\alpha + \left[A^i_\alpha, A_{\beta\dot\beta}\right] + \nabla^i_\beta A_{\alpha\dot\beta} - \nabla^i_{\alpha\dot\beta} A_\beta + \left[A^i_\beta, A_{\alpha\dot\beta}\right]\right),\\
F^i_{\dot\beta} &= \frac{1}{2}\epsilon^{\beta\alpha}\left(\nabla^i_\alpha A_{\beta\dot\beta} - \nabla_{\beta\dot\beta} A^i_\alpha + \left[A^i_\alpha, A_{\beta\dot\beta}\right]\right),\\
F^i_{\beta\dot\alpha\dot\beta} &= \frac{1}{2}\left(\nabla^i_{\dot\alpha} A_{\beta\dot\beta} - \nabla_{\beta\dot\beta} A^i_{\dot\alpha} + \left[A^i_{\dot\alpha}, A_{\beta\dot\beta}\right] + \nabla^i_{\dot\beta} A_{\beta\dot\alpha} - \nabla_{\beta\dot\alpha} A^i_{\dot\beta} + \left[A^i_{\dot\beta}, A_{\beta\dot\alpha}\right]\right),\\
F^i_\beta &= \frac{1}{2}\epsilon^{\dot\beta\dot\alpha}\left(\nabla^i_{\dot\alpha} A_{\beta\dot\beta} - \nabla_{\beta\dot\beta} A^i_{\dot\alpha} + \left[A^i_{\dot\alpha}, A_{\beta\dot\beta}\right]\right),\\
F_{\alpha\beta} &= \frac{1}{2}\epsilon^{\dot\beta\dot\alpha}\left(\nabla_{\alpha\dot\alpha} A_{\beta\dot\beta} - \nabla_{\beta\dot\beta} A_{\alpha\dot\alpha} + \left[A_{\alpha\dot\alpha}, A_{\beta\dot\beta}\right]\right),\\
F_{\dot\alpha\dot\beta} &= \frac{1}{2}\epsilon^{\beta\alpha}\left(\nabla_{\alpha\dot\alpha} A_{\beta\dot\beta} - \nabla_{\beta\dot\beta} A_{\alpha\dot\alpha} + \left[A_{\alpha\dot\alpha}, A_{\beta\dot\beta}\right]\right),
\end{aligned}
\tag{2.4}
$$

where we use the normalisation $\epsilon_{12} = \epsilon^{21} = 1$. It is obvious, in view of their construction that the covariant derivatives satisfy the generalised Jacobi identities which are nothing other than associativity conditions for the ∇'s and of the potentials (which we assume hold).

The level one identities arise from the associativity of $D^i_\alpha, D^j_{\dot\alpha}, D_{\beta\dot\beta}$:

$$
\begin{aligned}
&\delta^{ij}\left(\epsilon_{\alpha\beta} F_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta} F_{\alpha\beta}\right) - \epsilon_{\alpha\beta}\left\{D^j_{\dot\alpha}, F^i_{\dot\beta}\right\} - \epsilon_{\dot\alpha\dot\beta}\left\{D^i_\alpha, F^j_\beta\right\} -\\
&\qquad - \left[D_{\beta\dot\beta}, F^{ij}_{\alpha\dot\alpha}\right] - \left\{D^i_\alpha, F^j_{\beta\dot\alpha\dot\beta}\right\} - \left\{D^j_{\dot\alpha}, F^i_{\alpha\beta\dot\beta}\right\} = 0\,.
\end{aligned}
\tag{2.5}
$$

On the other hand, the level two identities arise from the associativity of $D^i_\alpha, D^j_\beta, D^k_{\dot\gamma}$ and of $D^i_{\dot\alpha}, D^j_{\dot\beta}, D^k_\gamma$:

$$
\begin{aligned}
&\left[D^i_\alpha, F^{jk}_{\beta\dot\gamma}\right] + \left[D^j_\beta, F^{ik}_{\alpha\dot\gamma}\right] + \left[D^k_{\dot\gamma}, F^{ij}_{\alpha\beta}\right] + \epsilon_{\alpha\beta}\left[D^k_{\dot\gamma}, F^{ij}\right] +\\
&\qquad + \delta^{jk} F^i_{\alpha\beta\dot\gamma} + \delta^{ik} F^j_{\alpha\beta\dot\gamma} + \delta^{jk}\epsilon_{\alpha\beta} F^i_{\dot\gamma} - \delta^{ik}\epsilon_{\alpha\beta} F^j_{\dot\gamma} = 0\,,
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
&\left[D^i_{\dot\alpha}, F^{kj}_{\gamma\dot\beta}\right] + \left[D^j_{\dot\beta}, F^{ki}_{\gamma\dot\alpha}\right] + \left[D^k_\gamma, F^{ij}_{\dot\alpha\dot\beta}\right] + \epsilon_{\dot\alpha\dot\beta}\left[D^k_\gamma, \overline{F}^{ij}\right] +\\
&\qquad + \delta^{jk} F^i_{\gamma\dot\alpha\dot\beta} + \delta^{ik} F^j_{\gamma\dot\alpha\dot\beta} + \delta^{jk}\epsilon_{\dot\alpha\dot\beta} F^i_\gamma - \delta^{ik}\epsilon_{\dot\alpha\dot\beta} F^j_\gamma = 0\,.
\end{aligned}
\tag{2.7}
$$

We now ask whether one can find a set of sufficient conditions on certain curvatures which imply the usual self-duality, which can be expressed as

$$
F_{\dot\alpha\dot\beta} = 0\,. \tag{2.8}
$$

Extracting the Lorentz irreducible part symmetric in $\dot\alpha,\dot\beta$ and skew in α,β from (2.5), we find

$$\begin{aligned}\delta^{ij}F_{\dot\alpha\dot\beta} &= \frac{1}{2}\left(\left\{D^j_{\dot\alpha},F^i_{\dot\beta}\right\}+\left\{D^j_{\dot\beta},F^i_{\dot\alpha}\right\}\right)+\\ &+\frac{1}{4}\epsilon^{\beta\alpha}\left(\left[D_{\beta\dot\beta},F^{ij}_{\alpha\dot\alpha}\right]+\left[D_{\beta\dot\alpha},F^{ij}_{\alpha\dot\beta}\right]\right)+\frac{1}{2}\epsilon^{\beta\alpha}\left\{D^i_\alpha,F^j_{\beta\dot\alpha\dot\beta}\right\}.\end{aligned} \tag{2.9}$$

Extracting the antisymmetric part in α,β (and in i,j) from (2.6) and the symmetric part in $\dot\alpha,\dot\beta$ (and in i,j) in (2.7) yields, respectively

$$\left(\delta^{jk}F^i_{\dot\gamma}-\delta^{ik}F^j_{\dot\gamma}\right)=\frac{1}{2}\epsilon^{\beta\alpha}\left(\left[D^j_\alpha,F^{ik}_{\beta\dot\gamma}\right]-\left[D^i_\alpha,F^{jk}_{\beta\dot\gamma}\right]\right)-\left[D^k_{\dot\gamma},F^{ij}\right] \tag{2.10}$$

and

$$\begin{aligned}\delta^{jk}F^i_{\alpha\dot\alpha\dot\beta}+\delta^{ik}F^j_{\alpha\dot\alpha\dot\beta} &= -\frac{1}{2}\left(\left[D^i_{\dot\alpha},F^{kj}_{\alpha\dot\beta}\right]+\left[D^j_{\dot\alpha},F^{ki}_{\alpha\dot\beta}\right]+\left[D^i_{\dot\beta},F^{kj}_{\alpha\dot\alpha}\right]+\left[D^j_{\dot\beta},F^{ki}_{\alpha\dot\alpha}\right]\right)-\\ &-\left[D^k_\alpha,F^{ij}_{\dot\alpha\dot\beta}\right].\end{aligned} \tag{2.11}$$

We are now in a position to answer the above question. Let us proceed in two steps. First we look for non-democratic systems of super self-duality conditions, with indices $i=1$ and $i=2$ playing a special role. Then, we generalise to some more democratic systems.

Non-democratic systems. Let us first look at these identities in a non-democratic way and take (2.9) for $i=j=1$. We see obviously the following level one implication

$$\left\{\text{System 1}\equiv\left\{F^{11}_{\alpha\dot\alpha}=0\,,\ F^1_{\dot\alpha}=0\,,\ F^1_{\alpha\dot\alpha\dot\beta}=0\right\}\right\}\Longrightarrow F_{\dot\alpha\dot\beta}=0\,. \tag{2.12}$$

In other words, the usual selfduality (2.8) is a consequence of the super selfduality conditions of System 1. Furthermore, it then follows that the right-hand sides of (2.9) are automatically all zero for any i,j since they are identities and the left hand side is zero in all cases. Remark that the first equation in System 1 is identical to the requirement that $A_{\alpha\dot\alpha}$ becomes a dependent quantity, namely,

$$A_{\alpha\dot\alpha}=\nabla^1_\alpha A^1_{\dot\alpha}+\nabla^1_{\dot\alpha}A^1_\alpha+\left\{A^1_\alpha,A^1_{\dot\alpha}\right\} \tag{2.13}$$

while the second and third equation become differential equations for the A^1's. Now, if we take (2.11) for $i=j=k=1$, we see, in an analogous way, that

$$\text{Implication a}:\left\{F^{11}_{\alpha\dot\alpha}=0\,,\ F^{11}_{\dot\alpha\dot\beta}=0\right\}\Longrightarrow F^1_{\alpha\dot\alpha\dot\beta}=0\,. \tag{2.14}$$

Finally, if we take (2.10) for $i=1, j=k=2$, we see that

$$\text{Implication b}:\left\{F^{12}_{\alpha\dot\alpha}=0\,,\ F^{22}_{\alpha\dot\alpha}=0\,,\ F^{12}=0\right\}\Longrightarrow F^1_{\dot\alpha}=0\,. \tag{2.15}$$

Hence, in System 1, the second and third condition can independently be replaced by the conditions given in Implication a and Implication b. Consequently we obtain three further level two systems which imply the usual selfduality, namely

$$\begin{aligned}
\text{System } 2 &\equiv \left\{F^{11}_{\alpha\dot\alpha}=0\,,\ F^{1}_{\dot\alpha}=0\,,\ F^{11}_{\dot\alpha\dot\beta}=0\right\},\\
\text{System } 3 &\equiv \left\{F^{11}_{\alpha\dot\alpha}=0\,,\ F^{12}_{\alpha\dot\alpha}=0\,,\ F^{22}_{\alpha\dot\alpha}=0\,,\ F^{12}=0\,,\ F^{1}_{\ \alpha\dot\alpha\dot\beta}=0\right\},\\
\text{System } 4 &\equiv \left\{F^{11}_{\alpha\dot\alpha}=0\,,\ F^{12}_{\alpha\dot\alpha}=0\,,\ F^{22}_{\alpha\dot\alpha}=0\,,\ F^{12}=0\,,\ F^{11}_{\dot\alpha\dot\beta}=0\right\}.
\end{aligned} \tag{2.16}$$

For all the four systems, the first equation $F^{11}_{\alpha\dot\alpha}=0$ implies that the vector potential $A_{\alpha\dot\alpha}$ is the dependent quantity (2.13). Any of the Systems 1–4 separately constitutes a coherent set of super selfduality requirements.

Clearly, this approach is highly non democratic; only the indices $i=1$ and 2 are taken into account. However, it is the basic algebra and in an essential way (2.1), which play the crucial role of extending, in a subtle way, the results to the other values of the indices.

Democratic systems. We can however try to find sets of sufficient conditions which are more democratic among the indices. Following the same arguments as used in the preceding case, we find from (2.9), summing over the indices $i=j$ to obtain democracy, that

$$\left\{\text{System } 5 \equiv \left\{\sum_i F^{ii}_{\alpha\dot\alpha}=0\,,\ F^{i}_{\dot\alpha}=0\ \forall i\,,\ F^{i}_{\ \alpha\dot\alpha\dot\beta}=0\ \forall i\right\}\right\} \Longrightarrow F_{\dot\alpha\dot\beta}=0\,. \tag{2.17}$$

Now, if we take (2.11) fixing i and summing over $j=k$, we see in an analogous way that

$$\text{Implication c}: \left\{F^{ij}_{\alpha\dot\alpha}=0\ \forall i,j\,,\ F^{ij}_{\dot\alpha\dot\beta}=0\ \forall i,j\right\} \Longrightarrow F^{i}_{\ \alpha\dot\alpha\dot\beta}=0\ \forall i\,. \tag{2.18}$$

Finally, if we take (2.10) fixing i and summing over $j=k$, we see that

$$\text{Implication d}: \left\{F^{ij}_{\alpha\dot\alpha}=0\ \forall i,j\,,\ F^{ij}=0\ \forall i,j\right\} \Longrightarrow F^{i}_{\dot\alpha}=0\ \forall i\,. \tag{2.19}$$

In System 5, the first and the second condition and/or the first and the third conditions can be replaced using Implication c and/or Implication d. This leads to the democratic second level systems

$$\begin{aligned}
\text{System } 6 &\equiv \left\{F^{ij}_{\alpha\dot\alpha}=0\ \forall i,j\,,\ F^{i}_{\dot\alpha}=0\ \forall i\,,\ F^{ij}_{\dot\alpha\dot\beta}=0\ \forall i,j\right\},\\
\text{System } 7 &\equiv \left\{F^{ij}_{\alpha\dot\alpha}=0\ \forall i,j\,,\ F^{ij}=0\ \forall i,j\,,\ F^{i}_{\ \alpha\dot\alpha\dot\beta}=0\ \forall i\right\},\\
\text{System } 8 &\equiv \left\{F^{ij}_{\alpha\dot\alpha}=0\ \forall i,j\,,\ F^{ij}=0\ \forall i,j\,,\ F^{ij}_{\dot\alpha\dot\beta}=0\ \forall i,j\right\}.
\end{aligned} \tag{2.20}$$

Let us make a few comments on some of these systems. Systems 6–8 clearly contain Systems 2–4 as subsets and hence are more restrictive. System 5 on the other hand, though containing many more conditions than the non-democratic systems may be a valuable alternative. In particular, the democratic form of the derived vector potential is

$$A_{\alpha\dot\alpha}=\sum_i\left(\nabla^i_\alpha A^i_{\dot\alpha}+\nabla^i_{\dot\alpha}A^i_\alpha+\{A^i_\alpha,A^i_{\dot\alpha}\}\right). \tag{2.21}$$

Sytem 8, together with its implications (2.17) and (2.18), imposed in (2.2), yields

$$\left\{D^i_\alpha, D^j_{\dot\alpha}\right\} = \delta^{ij} D_{\alpha\dot\alpha}\,, \qquad \left\{D^i_{\dot\alpha}, D^j_{\dot\beta}\right\} = \epsilon_{\dot\alpha\dot\beta}\overline{F}^{ij}\,, \qquad \left\{D^i_\alpha, D^j_\beta\right\} = F^{ij}_{\alpha\beta}\,,$$
$$\left[D^i_{\dot\alpha}, D_{\beta\dot\beta}\right] = \epsilon_{\dot\alpha\dot\beta} F^i_\beta\,, \qquad \left[D^i_\alpha, D_{\beta\dot\beta}\right] = F^i{}_{\alpha\beta\dot\beta}\,, \qquad \left[D_{\alpha\dot\alpha}, D_{\beta\dot\beta}\right] = \epsilon_{\dot\alpha\dot\beta} F_{\alpha\beta}\,. \quad (2.22)$$

This is a further form of the super-self duality equations (System 8). The remaining lower level curvatures appearing in (2.22) are sources for the self-dual field $F_{\alpha\beta}$. In a chiral superspace spanned by $(\nabla^i_{\dot\alpha}, \nabla_{\alpha\dot\alpha})$, these reduce to the well known conditions given by the three equations in the right-hand column of (2.22), which provide consistent irreducible supermultiplets for any N [2].

3. Case of $d = 8 \pmod 4$

In this section, we generalise the above 4-dimensional discussion to $d = 4n, (n \geq 2)$. For Spin$(4n)$, there are two inequivalent fundamental spinor representations of dimension $2^{(2n-1)}$, S^+ and S^- and we take $\Sigma = \{S^+, S^-\}$. The vector is contained in their tensor product. The curvatures are defined by

$$\begin{aligned}
\left\{D^{(S^+)}_B, D^{(S^-)}_{C'}\right\} &= C\left(S^+, B, S^-, C'; V, M\right) D^{(V)}_M + \\
&\quad + \sum_{W\in\{S^+\otimes S^-\}} C\left(S^+, B, S^-, C'; W, L\right) F^{(W)}_L\,,\\
\left\{D^{(S^+)}_B, D^{(S^+)}_C\right\} &= \sum_{U^+\in\{S^+\vee S^+\}} C\left(S^+, B, S^+, C; U^+, L\right) F^{(U^+)}_L\,,\\
\left\{D^{(S^-)}_{B'}, D^{(S^-)}_{C'}\right\} &= \sum_{U^-\in\{S^-\vee S^-\}} C\left(S^-, B', S^-, C'; U^-, L\right) F^{(U^-)}_L\,,\\
\left[D^{(S^+)}_B, D^{(V)}_M\right] &= \sum_{T^-\in\{S^+\otimes V\}} C\left(S^+, B, V, M; T^-, D\right) F^{(T^-)}_D\,,\\
\left[D^{(S^-)}_{B'}, D^{(V)}_M\right] &= \sum_{T^+\in\{S^-\otimes V\}} C\left(S^-, B', V, M; T^+, D\right) F^{(T^+)}_D\,,\\
\left[D^{(V)}_M, D^{(V)}_N\right] &= F^{(A)}_{MN}\,.
\end{aligned} \qquad (3.1)$$

From the tensor products in appendix A, we note that the representation spaces $U^\pm$ are p-forms with corresponding indices L taking the form of p skewsymmetrised vector indices. It should be remarked that if identical representations occur in $U^+, U^-, \ldots$, the corresponding curvature components need to be distinguished from each other. The further representations $T^\pm$ are summands in the tensor products (A.3) and (A.4), for rank $r = 2n \geq 4$, namely,

$$\begin{aligned}
T^-_1 &= R(\pi_1 + \pi_r)\,, \qquad & T^-_2 &= R(\pi_{r-1})\,,\\
T^+_1 &= R(\pi_1 + \pi_{r-1})\,, \qquad & T^+_2 &= R(\pi_r)\,.
\end{aligned} \qquad (3.2)$$

The first level Jacobi identities are those involving $\{D^{(S^+)}_B, D^{(S^-)}_{C'}, D^{(V)}_M\}$, while the second level Jacobis involve $\{D^{(S^+)}_A, D^{(S^-)}_{B'}, D^{(S^+)}_C\}$ and $\{D^{(S^+)}_A, D^{(S^-)}_{B'}, D^{(S^-)}_{C'}\}$.

Level 1 super self-duality. From the super Jacobi identity (1.11) between the operators $\{D_B^{(S^+)}, D_{C'}^{(S^-)}, D_Q^{(V)}\}$, upon multiplication, for example, by $C(A,NM;V,R,V,Q)C(V,R;S^+,B,S^-,C')$, the product of inverse Clebsch-Gordon coefficients, where NM are the antisymmetric indices of the adjoint representation A, and summation over R,Q,B,C' (we assume summation over repeated indices labeling the states of a representation), we obtain the identity

$$\begin{aligned} F_{NM}^{(A)} &= \frac{1}{2}\left(\left[D_M^{(V)}, F_N^{(V)}\right] - \left[D_N^{(V)}, F_M^{(V)}\right]\right) + \\ &+ \sum_{T^- \in \{S^+ \otimes V\}} \alpha_1(T^-)\, C\left(A,NM;S^-,C',T^-,D\right)\left\{D_{C'}^{(S^-)}, F_D^{(T^-)}\right\} + \\ &+ \sum_{T^+ \in \{S^- \otimes V\}} \alpha_2\left(T^+\right) C\left(A,NM;S^+,B,T^+,D\right)\left\{D_B^{(S^+)}, F_D^{(T^+)}\right\}. \end{aligned} \tag{3.3}$$

Here, from the representations $W \in \{S^+ \otimes S^-\}$ on the right-hand side of the first line of (1.11), only the vector contributes; the remaining terms do not contribute to the adjoint representation. (The three-form, $W = \wedge^3 V$, which according to (A.10) occurs for $d \geq 12$, also has a non-zero contribution to the adjoint, $C(A,NM;V,P,\wedge^3 V,L) \neq 0$ (see (A.11) and (A.12)), but the corresponding term is zero under our projection in (3.3)). The coefficients α_i incorporate recoupling coefficients, for example

$$\begin{aligned} &C(A,NM;V,K,V,Q)\; C\left(V,K;S^+,B,S^-,C'\right) C\left(S^+,B,V,Q;T^-,D\right) = \\ &\quad = \alpha_1(T^-)\; C\left(A,NM;S^-,C',T^-,D\right). \end{aligned} \tag{3.4}$$

We note that the coefficients $C(A,NM;T^\pm,D,S^\pm,B)$ are non zero for $T_1^\pm$ in virtue of (A.13) and (A.14) and for $T_2^\pm = S^\pm$ in virtue of (A.7), since the adjoint representation $R(\pi_2) = (010\ldots 0)$ is always contained in these decompositions for even rank $r \geq 4$.

We thus see that $F^{(V)}$ and $F^{(T^\pm)}$ determine $F^{(A)}$. Under H, the tensors above decompose into their irreducible pieces transforming under representations in $\rho_H(S^\pm), \rho_H(V), \rho_H(T^\pm)$ and $\rho_H(A)$. Now self-duality means that $F^{(A)}$, under H-decomposition, is restricted to its components in $\rho_H(\lambda)$. We want to determine sets of sufficient conditions on certain pieces of $F^{(V)}$ and $F^{(T^\pm)}$, which imply that $F^{(A)}$ is restricted to live in a specific eigenspace $\rho_H(\lambda)$ (1.5). In order to ensure this, we need that the contributions to the complement $\rho_H(\lambda)^{\complement}$ from the curvature components on the right vanish.

Generically, let us consider a supercommutator involving $D^{(Y)}$ and $F^{(Z)}$, which produces the adjoint A (as in (3.3)). Let us define for representations Y and Z such that $A \subset Y \otimes Z$,

- the *source subset* $\sigma_{H,\lambda}(Y,Z) \subset \rho_H(Z)$,

$$\sigma_{H,\lambda}(Y,Z) := \left\{ \underline{z} \in \rho_H(Z) \,\middle|\, \left(\bigcup_{\underline{y} \in \rho_H(Y)} \{\underline{y} \otimes \underline{z}\} \right) \bigcap \rho_H(\lambda) \neq \emptyset \right\} \tag{3.5}$$

- the *sink subset* $\overset{\triangledown}{\sigma}_{H,\lambda}(Y,Z) \subset \rho_H(Z)$,

$$\overset{\triangledown}{\sigma}_{H,\lambda}(Y,Z) := \left\{ \underline{z} \in \rho_H(Z) \middle| \left(\bigcup_{\underline{y} \in \rho_H(Y)} \{\underline{y} \otimes \underline{z}\} \right) \bigcap \rho_H(\lambda)^{\complement} \neq \emptyset \right\}. \tag{3.6}$$

Here $\{\underline{y} \otimes \underline{z}\}$ denotes the union of all irreducible H-representations contained in the tensor product $\underline{y} \otimes \underline{z}$. We now see from (3.3) that the supercurvatures corresponding to the *source-subsets* σ, $\{F^{(V)}(\underline{v}),\ F^{(T^\pm)}(\underline{t}^\pm);\ \underline{w} \in \sigma_{H,\lambda}(V,V),\ \underline{t}^\pm \in \sigma_{H,\lambda}(S^\pm,T^\pm)\}$, yield contributions to $\{F^{(A)}(\underline{a});\ \underline{a} \in \rho_H(\lambda)\}$, i.e. to the parts of the curvature which do not vanish in (1.5). On the other hand, the supercurvatures corresponding to the *sink-subsets* $\overset{\triangledown}{\sigma}$, $\{F^{(V)}(\underline{v}),\ F^{(T^\pm)}(\underline{t}^\pm);\ \underline{v} \in \overset{\triangledown}{\sigma}_{H,\lambda}(V,V),\ \underline{t}^\pm \in \overset{\triangledown}{\sigma}_{H,\lambda}(S^\pm,T^\pm)\}$, yield contributions to $\{F^{(A)}(\underline{a})\,;\ \underline{a} \in \rho_H(\lambda)^{\complement}\}$, i.e. to those parts of the curvature which appear in the conditions (1.6). Thus, the conditions (1.6) are implied by the following level one supercurvature constraints:

$$F^{(V)}(\underline{v}) = 0\,, \qquad \text{for all } \underline{v} \in \overset{\triangledown}{\sigma}_{H,\lambda}(V,V)\,, \tag{3.7}$$

$$F^{(T^+)}\left(\underline{t}^+\right) = 0\,, \qquad \text{for all } \underline{t}^+ \in \overset{\triangledown}{\sigma}_{H,\lambda}\left(S^+,T^+\right), \tag{3.8}$$

$$F^{(T^-)}\left(\underline{t}^-\right) = 0\,, \qquad \text{for all } \underline{t}^- \in \overset{\triangledown}{\sigma}_{H,\lambda}(S^-,T^-)\,. \tag{3.9}$$

We remark that if the vector is irreducible under H, i.e. $\rho_H(V) = \{\underline{v}\}$, then $\overset{\triangledown}{\sigma}_{H,\lambda}(V,V) = \rho_H(V)$ and we need to impose $F^{(V)}(\underline{v}) = 0$ for all λ's. In order to have non-trivial self-duality (1.1), we further need to check that the imposition of (3.7), (3.8) and (3.9) does not imply the vanishing of all the $F^{(A)}(\underline{a})$'s. In particular, at least one of the components of the curvature in (1.5) needs to be non zero. In other words, we require that

$$F^{(A)}(\underline{a}) \neq 0\,, \qquad \text{for at least one } \underline{a} \in \rho_H(\lambda)\,. \tag{3.10}$$

This follows if a piece of any of the source-subsets $\sigma_{H,\lambda}(V,V)$, $\sigma_{H,\lambda}(S^+,T^+)$ or $\sigma_{H,\lambda}(S^-,T^-)$ lies, respectively, in the complement of the corresponding sink-subsets $\rho_H(V)\backslash\overset{\triangledown}{\sigma}_{H,\lambda}(V,V)$ or $\rho_H(T^+)\backslash\overset{\triangledown}{\sigma}_{H,\lambda}(S^+,T^+)$ or $\rho_H(T^-)\backslash\overset{\triangledown}{\sigma}_{H,\lambda}(S^-,T^-)$. We define, for representations Y and Z such that $A \subset Y \otimes Z$,

- the *wet source subset* $\breve{\sigma}_{H,\lambda}(Y,Z)$ composed of the source representations not contained in the sink-subset,

$$\begin{aligned} \breve{\sigma}_{H,\lambda}(Y,Z) &:= \sigma_{H,\lambda}(Y,Z) \cap \left(\rho_H(Z)\backslash\overset{\triangledown}{\sigma}_{H,\lambda}(Y,Z)\right) \\ &= \sigma_{H,\lambda}(Y,Z)\backslash\left(\sigma_{H,\lambda}(Y,Z) \cap \overset{\triangledown}{\sigma}_{H,\lambda}(Y,Z)\right). \end{aligned} \tag{3.11}$$

The condition for nontriviality is that the corresponding set of supercurvature components,

$$\left\{F^{(V)}(\underline{v})\,,\ F^{(T^\pm)}\left(\underline{t}^\pm\right) \middle| \underline{v} \in \breve{\sigma}_{H,\lambda}(V,V)\,,\ \underline{t}^\pm \in \breve{\sigma}_{H,\lambda}\left(S^\pm,T^\pm\right)\right\}, \tag{3.12}$$

is non empty. These act as sources for the non-zero fields $F^{(A)}(\underline{a})$ in (3.10). If this set turns out to be empty, eqs. (3.7), (3.8) and (3.9) are too strong, implying flatness: $F^{(A)}(\underline{a}) = 0$ for all $\underline{a}$.

We note that the level one conditions (3.7), (3.8) and (3.9) replace the maximal set of equations $F^{(A)}(\underline{a}) = 0$, for every $\underline{a} \in \rho_H(\lambda)^{\complement}$. Possibilities in which some, but not all, of these curvature constraints are replaced by some level one (first order) constraints (without implying full flatness) clearly yield alternative sets of sufficient conditions implying (1.1). For cases in which the maximal replacement (3.7), (3.8) and (3.9) yields complete flatness, such non-maximal possibilities (when they are allowed) yield alternative sets of super self-dualities. A further alternative possibility is the higher-order one, in which the required conditions (3.10) are obtained when some of the zero curvature conditions $F^{(T^{\pm})}(\underline{t}^{\pm}) = 0$ in (3.8) and (3.9) are replaced by the corresponding chirality conditions of the form $\{D_{B'}^{(S^{\pm})}(\underline{s}^{\pm}), F^{(T^{\pm})}(\underline{t}^{\pm})\} = 0$, (second order in derivatives) for specific choices of $\underline{s}^{\pm}, \underline{t}^{\pm}$. We shall discuss explicit examples of both these alternative possibilities for various examples in which the maximal replacement is tantamount to complete flatness.

Level 2 super self-duality. The second level Jacobi identity among $\{D_B^{(S^+)}$, $D_{B'}^{(S^-)}$, $D_C^{(S^+)}\}$ takes the bare form

$$\begin{aligned}
&\left(C\left(S^+,B,S^-,B';V,M\right)C\left(S^+,C,V,M;T^-,D\right)+\right.\\
&\left.+C\left(S^+,C,S^-,B';V,M\right)C\left(S^+,B,V,M;T^-,D\right)\right)F_D^{(T^-)}=\\
&=-\sum_{W\in\{S^+\otimes S^-\}}C\left(S^+,B,S^-,B';W,Q\right)\left[D_C^{(S^+)},F_Q^{(W)}\right]-\\
&-\sum_{W\in\{S^+\otimes S^-\}}C\left(S^+,C,S^-,B';W,Q\right)\left[D_B^{(S^+)},F_Q^{(W)}\right]-\\
&-\sum_{U^+\in\{S^+\otimes S^+\}}C\left(S^+,B,S^+,C;U^+,Q\right)\left[D_{B'}^{(S^-)},F_Q^{(U^+)}\right]
\end{aligned}\tag{3.13}$$

and similarly, the associativity of $\{D_B^{(S^+)}$, $D_{B'}^{(S^-)}$, $D_{C'}^{(S^-)}\}$ yields the bare identity

$$\begin{aligned}
&\left(C\left(S^+,B,S^-,B';V,M\right)C\left(S^-,C',V,M;T^+,D\right)+\right.\\
&\left.+C\left(S^+,B,S^-,C';V,M\right)C\left(S^-,B',V,M;T^+,D\right)\right)F_D^{(T^+)}=\\
&=-\sum_{W\in\{S^+\otimes S^-\}}C\left(S^+,B,S^-,B';W,Q\right)\left[D_{C'}^{(S^-)},F_Q^{(W)}\right]-\\
&-\sum_{W\in\{S^+\otimes S^-\}}C\left(S^+,B,S^-,C';W,Q\right)\left[D_{B'}^{(S^-)},F_Q^{(W)}\right]-\\
&-\sum_{U^-\in\{S^-\otimes S^-\}}C\left(S^-,B',S^-,C';U^-,Q\right)\left[D_B^{(S^+)},F_Q^{(U^-)}\right].
\end{aligned}\tag{3.14}$$

Using properties of the Clebsch-Gordon coefficients we may isolate $F_E^{(T^+)}$ and $F_E^{(T^-)}$ in the forms,

$$\begin{aligned}
F_E^{(T^+)}=&\sum_{W\in\{S^+\otimes S^-\}}\alpha_3(W)C\left(T^+,E;S^-,B',W,Q\right)\left[D_{B'}^{(S^-)},F_Q^{(W)}\right]+\\
&+\sum_{U^-\in\{S^-\otimes S^-\}}\alpha_4(U^-)C\left(T^+,E;S^+,B,U^-,Q\right)\left[D_B^{(S^+)},F_Q^{(U^-)}\right],
\end{aligned}\tag{3.15}$$

and

$$
\begin{aligned}
F_E^{(T^-)} &= \sum_{W\in\{S^+\otimes S^-\}} \alpha_5(W) C\left(T^-, E; S^+, B, W, Q\right)\left[D_B^{(S^+)}, F_Q^{(W)}\right] + \\
&\quad + \sum_{U^+\in\{S^+\otimes S^+\}} \alpha_6\left(U^+\right) C\left(T^-, E; S^-, B', U^+, Q\right)\left[D_{B'}^{(S^-)}, F_Q^{(U^+)}\right], \quad (3.16)
\end{aligned}
$$

where the α_i depend on appropriate recoupling coefficients. We thus see that the possibility exists of making the conditions (3.8) and (3.9) automatic in virtue of appropriate conditions on $F_Q^{(W)}(\underline{w})$, $F_Q^{(U^+)}(\underline{u}^+)$ and $F_Q^{(U^-)}(\underline{u}^-)$.

Suppose that the supercommutator of $D^{(Y)}$ with $F^{(Z)}$ manufactures a curvature component $F^{(X)}$. Further, suppose that the supercommutator of $D^{(K)}$ with this $F^{(X)}$ contributes to the adjoint A (as in (3.3)). For representations K, X, Y, Z such that $X \subset Y \otimes Z$ (as with e.g. $T^+ \subset S^- \otimes W$ in (3.15)) and $A \subset K \otimes X$, we define further natural source and sink subsets $\tau_{H,\lambda}(K, X; Y, Z)$, $\overset{\triangledown}{\tau}_{H,\lambda}(K, X; Y, Z) \subset \rho_H(K)$ by

$$
\begin{aligned}
\tau_{H,\lambda}(K,X;Y,Z) &:= \left\{ \underline{z} \in \rho_H(Z) \,\middle|\, \left(\bigcup_{\underline{y}\in\rho_H(Y)} \{\underline{y}\otimes\underline{z}\} \right) \bigcap \sigma_{H,\lambda}(K,X) \neq \emptyset \right\}, \\
\overset{\triangledown}{\tau}_{H,\lambda}(K,X;Y,Z) &:= \left\{ \underline{z} \in \rho_H(Z) \,\middle|\, \left(\bigcup_{\underline{y}\in\rho_H(Y)} \{\underline{y}\otimes\underline{z}\} \right) \bigcap \overset{\triangledown}{\sigma}_{H,\lambda}(K,X) \neq \emptyset \right\}. \quad (3.17)
\end{aligned}
$$

We denote by $\overset{\smile}{\tau}$ the wet sources, the intersection of the source subset τ with the complement of the corresponding sink subset $\overset{\triangledown}{\tau}$,

$$
\overset{\smile}{\tau}_{H,\lambda}(K,X;Y,Z) := \tau_{H,\lambda}(K,X;Y,Z) \cap \left(\rho_H(K) \backslash \overset{\triangledown}{\tau}_{H,\lambda}(K,X;Y,Z)\right). \quad (3.18)
$$

From (3.16) we see that in order to have (3.8), it suffices to impose

$$
\begin{aligned}
F^{(W)}(\underline{w}) &= 0\,, \qquad \text{for all } \underline{w} \in \overset{\triangledown}{\tau}_{H,\lambda}\left(S^+, T^+; S^-, W\right), \\
F^{(U^-)}(\underline{u}^-) &= 0\,, \qquad \text{for all } \underline{u}^- \in \overset{\triangledown}{\tau}_{H,\lambda}\left(S^+, T^+; S^+, U^-\right). \quad (3.19)
\end{aligned}
$$

Similarly, we see from (3.15) that in order to have (3.9), it suffices to impose

$$
\begin{aligned}
F^{(W)}(\underline{w}) &= 0\,, \qquad \text{for all } \underline{w} \in \overset{\triangledown}{\tau}_{H,\lambda}\left(S^-, T^-; S^+, W\right), \\
F^{(U^+)}\left(\underline{u}^+\right) &= 0\,, \qquad \text{for all } \underline{u}^+ \in \overset{\triangledown}{\tau}_{H,\lambda}\left(S^-, T^-; S^-, U^+\right). \quad (3.20)
\end{aligned}
$$

We note from (A.10), (A.5) and (A.6) that

$$
\begin{aligned}
W &\in \left\{ R(\pi_r + \pi_{r-1})\,,\ R(\pi_{2p+1})\,;\ 0 \le p \le \frac{r-4}{2} \right\}, \\
U^+ &\in \left\{ R(2\pi_r)\,,\ R(\pi_{r-4p})\,;\ 1 \le p \le \left[\frac{r}{4}\right] \right\}, \\
U^- &\in \left\{ R(2\pi_{r-1})\,,\ R(\pi_{r-4p})\,;\ 1 \le p \le \left[\frac{r}{4}\right] \right\}. \quad (3.21)
\end{aligned}
$$

From the tensor products (A.18) and (A.19) relevant for (3.15), we see that $U_r^- \otimes S^+ = R(2\pi_{r-1}) \otimes R(\pi_r)$ contains T_1^+ but not T_2^+. Thus $\tau_{H,\lambda}(S^+, T_2^+; S^+, U_r^-) = \overset{\triangledown}{\tau}_{H,\lambda}(S^+, T_2^+; S^+, U_r^-) = \emptyset$. When the rank $r = 4 \pmod 4$, there exists a scalar amongst the U^- and obviously $R(0) \otimes R(\pi_r)$ contains T_2^+ but not T_1^+. All the other tensor products (A.18) and (A.19)contain both T_1^+ and T_2^+, so that the corresponding $\tau, \overset{\triangledown}{\tau}$ are a priori not empty sets. Similarly, the tensor products relvant for (3.16), namely (A.15)–(A.22), show that $U_r^+ \otimes S^- = R(2\pi_r) \otimes R(\pi_{r-1})$ contains T_1^- but not T_2^-, yielding $\tau_{H,\lambda}(S^-, T_2^-; S^-, U_r^+) = \overset{\triangledown}{\tau}_{H,\lambda}(S^-, T_2^-; S^-, U_r^+) = \emptyset$. All the other decompositions (A.15)–(A.22) contain both T_1^- and T_2^-, except when the rank $r = 4 \pmod 4$, when the scalar amongst the U^+ does not yields T_1^-.

In order to have non-trivial super self-duality conditions, we need to check that imposing the conditions (3.19) and (3.20) leaves, respectively,

$$F^{(T^+)}\left(\underline{t}^+\right) \neq 0\,, \qquad \text{for at least one } \underline{t}^+ \in \sigma_{H,\lambda}\left(S^+, T^+\right) \tag{3.22}$$

and

$$F^{(T^-)}(\underline{t}^-) \neq 0\,, \qquad \text{for at least one } \underline{t}^- \in \sigma_{H,\lambda}(S^-, T^-)\,. \tag{3.23}$$

The former condition follows if we have either

$$\breve{\tau}_{H,\lambda}\left(S^+, T^+; S^-, W\right) \neq \emptyset\,, \qquad \text{or} \qquad \breve{\tau}_{H,\lambda}\left(S^+, T^+; S^+, U^-\right) \neq \emptyset\,. \tag{3.24}$$

Similarly, (3.23) follows if we have either

$$\breve{\tau}_{H,\lambda}\left(S^-, T^-; S^+, W\right) \neq \emptyset\,, \qquad \text{or} \qquad \breve{\tau}_{H,\lambda}\left(S^-, T^-; S^-, U^+\right) \neq \emptyset\,. \tag{3.25}$$

Summarising, we see that the set of eqs. {(3.7), (3.8) and (3.9)} provide a system of sufficient conditions implying the self-duality eqs. (1.6). In this set, replacing (3.8) by (3.19) and/or (3.9) by (3.20) yields further sets of sufficient conditions for (1.6).

We now discuss some explicit examples in $d = 8$.

3.1 $H = \mathrm{Spin}(7) \subset \mathrm{Spin}(8)$

Spin(8) has three 8-dimensional representations, which we assign as: $V = (1000)$, $S^+ = (0001)$ and $S^- = (0010)$. The representations occuring in (3.1), taking into account the appropriate symmetry or skewsymmetry property, are given by

$$\begin{aligned}
W &\in \left\{S^+ \otimes S^-\right\} = \left\{W_3 = (0011)_{\mathbf{56_V}}\,,\ W_1 = V = (1000)_{\mathbf{8_V}}\right\},\\
U^+ &\in \left\{S^+ \vee S^+\right\} = \left\{U_4^+ = (0002)_{\mathbf{35_{S^+}}}\,,\ U_0^+ = (0000)_{\mathbf{1}}\right\},\\
U^- &\in \left\{S^- \vee S^-\right\} = \left\{U_4^- = (0020)_{\mathbf{35_{S^-}}}\,,\ U_0^- = (0000)_{\mathbf{1}}\right\},\\
T^+ &\in \left\{S^- \otimes V\right\} = \left\{T_1^+ = (1010)_{\mathbf{56_{S^+}}}\,,\ T_2^+ = S^+ = (0001)_{\mathbf{8_{S^+}}}\right\},\\
T^- &\in \left\{S^+ \otimes V\right\} = \left\{T_1^- = (1001)_{\mathbf{56_{S^-}}}\,,\ T_2^- = S^- = (0010)_{\mathbf{8_{S^-}}}\right\},\\
A &= V \wedge V = (0100)_{\mathbf{28}}\,,
\end{aligned} \tag{3.26}$$

where, for convenience, we append the dimension of the representation to the Dynkin indices. From the decompositions of the above representations into irreducible Spin(7) representations, we obtain the set of supercurvature components $F^{(X)}(\underline{x})$, with the possible values of the Spin(7) representations $\underline{x} = \underline{w}, \underline{v}, \underline{u}^+, \underline{u}^-, \underline{t}^+, \underline{t}^-, \underline{a}$, being given by,

X	$\rho_{\mathrm{Spin}\,7}(X)$
$W_3 = (0011)_{\mathbf{56_V}}$	$\{\underline{w}_{31} = (101)_{\underline{\mathbf{48}}}\,,\ \underline{w}_{32} = (001)_{\underline{\mathbf{8}}}\}$
$W_1 = V = (1000)_{\mathbf{8}}$	$\{\underline{w}_1 = \underline{v} = (001)_{\underline{\mathbf{8}}}\}$
$U_4^+ = (0002)_{\mathbf{35_{S^+}}}$	$\{\underline{u}_{41}^+ = (200)_{\underline{\mathbf{27}}}\,,\ \underline{u}_{42}^+ = (100)_{\underline{\mathbf{7}}}\,,\ \underline{u}_{43}^+ = (000)_{\underline{\mathbf{1}}}\}$
$U_0^+ = (0000)_{\mathbf{1}}$	$\{\underline{u}_0^+ = (000)_{\underline{\mathbf{1}}}\}$
$U_4^- = (0020)_{\mathbf{35_{S^-}}}$	$\{\underline{u}_4^- = (002)_{\underline{\mathbf{35}}}\}$
$U_0^- = (0000)_{\mathbf{1}}$	$\{\underline{u}_0^- = (000)_{\underline{\mathbf{1}}}\}$
$T_1^+ = (1010)_{\mathbf{56_{S^+}}}$	$\{\underline{t}_{11}^+ = (002)_{\underline{\mathbf{35}}}\,,\ \underline{t}_{12}^+ = (010)_{\underline{\mathbf{21}}}\}$
$T_2^+ = S^+ = (0001)_{\mathbf{8_{S^+}}}$	$\{\underline{t}_{21}^+ = \underline{s}_1^+ = (100)_{\underline{\mathbf{7}}}\,,\ \underline{t}_{22}^+ = \underline{s}_2^+ = (000)_{\underline{\mathbf{1}}}\}$
$T_1^- = (1001)_{\mathbf{56_{S^-}}}$	$\{\underline{t}_{11}^- = (101)_{\underline{\mathbf{48}}}\,,\ \underline{t}_{12}^- = (001)_{\underline{\mathbf{8}}}\}$
$T_2^- = S^- = (0010)_{\mathbf{8_{S^-}}}$	$\{\underline{t}_2^- = \underline{s}^- = (001)_{\underline{\mathbf{8}}}\}$
$A = (0100)_{\mathbf{28}}$	$\{\underline{a}_1 = (010)_{\underline{\mathbf{21}}}\,,\ \underline{a}_2 = (100)_{\underline{\mathbf{7}}}\}$

The curvature components $F^{(A)}(\underline{a}_1)$ and $F^{(A)}(\underline{a}_2)$ form the two eigenspaces, with eigenvalues $\lambda = 1, -3$, respectively, of the Spin(7)-invariant T-tensor corresponding to the representation $\underline{u}_{43}^+$ above. In an explicit coordinate system, this T-tensor, as well as the two sets of self-duality equations, are displayed in [1]. The tensor products which contribute to the adjoint in (3.3) are $V \otimes V$ and $S^\pm \otimes T^\pm$, and those which contribute to $T^\pm$ in (3.15) and (3.16) are $S^\mp \otimes W$ and $S^\pm \otimes U^\mp$. The (non-trivial) Spin(7) tensor products which descend from these are,

$$\begin{aligned} \underline{s}^- \otimes \underline{t}_{11}^- &= \underline{s}^- \otimes \underline{w}_{31} = (001)_{\underline{\mathbf{8}}} \otimes (101)_{\underline{\mathbf{48}}} \\ &= (102)_{\underline{\mathbf{189}}} \oplus (110)_{\underline{\mathbf{105}}} \oplus (002)_{\underline{\mathbf{35}}} \oplus (200)_{\underline{\mathbf{27}}} \oplus (010)_{\underline{\mathbf{21}}} \oplus (100)_{\underline{\mathbf{7}}}\,, \end{aligned} \tag{3.27}$$

$$\begin{aligned} \underline{s}^- \otimes \underline{t}_{12}^- &= \underline{s}^- \otimes \underline{t}_2^- = \underline{v} \otimes \underline{v} = \underline{s}^- \otimes \underline{w}_{32} = \underline{s}^- \otimes \underline{w}_1 = (001)_{\underline{\mathbf{8}}} \otimes (001)_{\underline{\mathbf{8}}} \\ &= (002)_{\underline{\mathbf{35}}_s} \oplus (010)_{\underline{\mathbf{21}}_a} \oplus (100)_{\underline{\mathbf{7}}_a} \oplus (000)_{\underline{\mathbf{1}}_s}\,, \end{aligned} \tag{3.28}$$

$$\underline{s}^- \otimes \underline{u}_{41}^+ = (001)_{\underline{\mathbf{8}}} \otimes (200)_{\underline{\mathbf{27}}} = (201)_{\underline{\mathbf{168}}} \oplus (101)_{\underline{\mathbf{48}}}\,, \tag{3.29}$$

$$\underline{s}^- \otimes \underline{u}_{42}^+ = \underline{w}_{32} \otimes \underline{s}_1^+ = \underline{w}_1 \otimes \underline{s}_1^+ = (001)_{\underline{\mathbf{8}}} \otimes (100)_{\underline{\mathbf{7}}} = (101)_{\underline{\mathbf{48}}} \oplus (001)_{\underline{\mathbf{8}}}\,, \tag{3.30}$$

$$\underline{s}_1^+ \otimes \underline{w}_{31} = (100)_{\underline{\mathbf{7}}} \otimes (101)_{\underline{\mathbf{48}}} = (201)_{\underline{\mathbf{168}}} \oplus (011)_{\underline{\mathbf{112}}} \oplus (101)_{\underline{\mathbf{48}}} \oplus (001)_{\underline{\mathbf{8}}}\,, \tag{3.31}$$

$$\underline{s}_1^+ \otimes \underline{t}_{11}^+ = \underline{s}_1^+ \otimes \underline{u}_4^- = (100)_{\underline{\mathbf{7}}} \otimes (002)_{\underline{\mathbf{35}}} = (102)_{\underline{\mathbf{189}}} \oplus (002)_{\underline{\mathbf{35}}} \oplus (010)_{\underline{\mathbf{21}}}\,, \tag{3.32}$$

$$\underline{s}_1^+ \otimes \underline{t}_{12}^+ = (100)_{\underline{\mathbf{7}}} \otimes (010)_{\underline{\mathbf{21}}} = (110)_{\underline{\mathbf{105}}} \oplus (002)_{\underline{\mathbf{35}}} \oplus (100)_{\underline{\mathbf{7}}}\,, \tag{3.33}$$

$$\underline{s}_1^+ \otimes \underline{t}_{21}^+ = (100)_{\underline{\mathbf{7}}} \otimes (100)_{\underline{\mathbf{7}}} = (200)_{\underline{\mathbf{27}}_s} \oplus (010)_{\underline{\mathbf{21}}_a} \oplus (000)_{\underline{\mathbf{1}}_s}\,, \tag{3.34}$$

$$\underline{s}_1^+ \otimes \underline{t}_{22}^+ = \underline{s}_2^+ \otimes \underline{t}_{21}^+ = \underline{s}_1^+ \otimes \underline{u}_0^- = (100)_{\underline{\mathbf{7}}}\,, \qquad \underline{s}_2^+ \otimes \underline{t}_{12}^+ = (010)_{\underline{\mathbf{21}}}\,, \tag{3.35}$$

$$\underline{s}_2^+ \otimes \underline{w}_{32} = \underline{s}_2^+ \otimes \underline{w}_1 = \underline{s}^- \otimes \underline{u}_{43}^+ = \underline{s}^- \otimes \underline{u}_0^+ = (001)_{\underline{\mathbf{8}}}\,, \qquad \underline{s}_2^+ \otimes \underline{w}_{31} = (101)_{\underline{\mathbf{48}}}\,, \tag{3.36}$$

$$\underline{s}_2^+ \otimes \underline{t}_{11}^+ = \underline{s}_2^+ \otimes \underline{u}_4^- = (002)_{\underline{\mathbf{35}}}\,, \qquad \underline{s}_2^+ \otimes \underline{u}_0^- = \underline{s}_2^+ \otimes \underline{t}_{22}^+ = (000)_{\underline{\mathbf{1}}}\,. \tag{3.37}$$

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We note that since the vector representation V, under which both $F^{(V)}$ and $D^{(V)}$ in (3.3) transform, remains irreducible under Spin(7), the product $\underline{v}\otimes\underline{v}$ in (3.28) contains the entire adjoint representation $A = \underline{a}_1 \oplus \underline{a}_2$. Hence, $\overset{\triangledown}{\sigma}_{\mathrm{Spin}(7),\lambda}(V,V) = \{\underline{v}\} = \rho_{\mathrm{Spin}(7)}(V)$ for both values of λ, yielding, according to (3.7), the first part of the superduality system,

$$F^{(V)}(\underline{v}) = 0\,. \tag{3.38}$$

Similarly, from (3.27) and (3.28), we see that the tensor products contributing to the second line of (3.3) also contain both $\underline{a}_1$ and $\underline{a}_2$ parts of the adjoint. We therefore have, for both values of λ, $\overset{\triangledown}{\sigma}_{\mathrm{Spin}(7),\lambda}(S^-,T^-) = \rho_{\mathrm{Spin}(7)}(T^-)$ yielding the relations,

$$F^{(T_1^-)}\left(\underline{t}_{11}^-\right) = F^{(T_1^-)}\left(\underline{t}_{12}^-\right) = F^{(T_2^-)}\left(\underline{t}_2^-\right) = 0\,. \tag{3.39}$$

$\boldsymbol{\lambda_{\underline{21}} = 1.}$ For this eigenvalue, self-duality is given by the seven equations

$$F^{(A)}\left(\underline{a}_2\right) \equiv F^{(0100)}\left((100)_{\underline{\mathbf{7}}}\right) = 0\,, \qquad F^{(A)}\left(\underline{a}_1\right) \equiv F^{(0100)}\left((010)_{\underline{\mathbf{21}}}\right) \neq 0\,. \tag{3.40}$$

We are now in a position to read off the remaining level one conditions (3.8). The set of Spin(7) tensor products (3.32)–(3.35) shows that

$$\underline{a}_2 = (100)_{\underline{\mathbf{7}}} \in \left\{ \bigcup_{\underline{s}^+} \left\{\underline{s}^+ \otimes \underline{t}^+\right\} \right\}, \qquad \text{for } \underline{t}^+ = \underline{t}_{12}^+, \underline{t}_{21}^+, \underline{t}_{22}^+\,, \tag{3.41}$$

but not for $\underline{t}^+ = \underline{t}_{11}^+$. We therefore have

$$\begin{aligned} \overset{\triangledown}{\sigma}_{\mathrm{Spin}(7),\lambda=1}\left(S^+,T^+\right) &= \left\{\underline{t}_{12}^+, \underline{t}_{21}^+, \underline{t}_{22}^+\right\}, \\ \breve{\sigma}_{\mathrm{Spin}(7),\lambda=1}\left(S^+,T^+\right) &= \left\{\underline{t}_{11}^+\right\}. \end{aligned} \tag{3.42}$$

Thus, the set of constraints, which together with (3.38) and (3.39), form the level one super self-duality equations for $\lambda = 1$ are

$$F^{(T_1^+)}\left(\underline{t}_{12}^+\right) = F^{(T_2^+)}\left(\underline{t}_{21}^+\right) = F^{(T_2^+)}\left(\underline{t}_{22}^+\right) = 0\,, \qquad F^{(T_1^+)}\left(\underline{t}_{11}^+\right) \neq 0\,. \tag{3.43}$$

Moreover, in virtue of the first tensor product in (3.37), the latter component provides a wet source for $F^{(A)}(\underline{a}_1)$, the 21-dimensional part of $F^{(A)}$, which is required to be non zero.

Proceeding in the same way to the level two identities (3.15) and (3.16) and recalling the definition (3.17), we find, using (3.27)–(3.37), that,

$$\begin{aligned} \overset{\triangledown}{\tau}_{\mathrm{Spin}(7),\lambda=1}\left(S^\pm,T^\pm;S^\mp,W_i\right) &= \rho_{\mathrm{Spin}(7)}(W_i)\,, \qquad i = 1,3\,, \\ \overset{\triangledown}{\tau}_{\mathrm{Spin}(7),\lambda=1}\left(S^\pm,T^\pm;S^\pm,U_i^\mp\right) &= \rho_{\mathrm{Spin}(7)}(U_i^-)\,, \qquad i = 0,4\,. \end{aligned} \tag{3.44}$$

There are therefore no level two wet sources and no non-trivial level two super self-dualities.

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$\boldsymbol{\lambda_{\underline{7}} = -3.}$ For this eigenvalue, self-duality is given by the 21 equations

$$F^{(A)}(\underline{a}_1) \equiv F^{(0100)}\left((010)_{\underline{\mathbf{21}}}\right) = 0\,, \qquad F^{(A)}(\underline{a}_2) \equiv F^{(0100)}\left((100)_{\underline{\mathbf{7}}}\right) \neq 0\,. \tag{3.45}$$

Now, from (3.32)–(3.35) we see that

$$\underline{a}_1 = (010)_{\underline{\mathbf{21}}} \in \left\{ \bigcup_{\underline{s}^+} \{\underline{s}^+ \otimes \underline{t}^+\} \right\}, \qquad \text{for } \underline{t}^+ = \underline{t}^+_{11}, \underline{t}^+_{12}, \underline{t}^+_{21}\,. \tag{3.46}$$

Therefore,

$$\begin{aligned} \overset{\triangledown}{\sigma}_{\mathrm{Spin}(7),\lambda=-3}\left(S^+, T^+\right) &= \{\underline{t}^+_{11}, \underline{t}^+_{12}, \underline{t}^+_{21}\}\,, \\ \breve{\sigma}_{\mathrm{Spin}\,7,\lambda=-3}\left(S^+, T^+\right) &= \{\underline{t}^+_{22}\}\,. \end{aligned} \tag{3.47}$$

This yields the conditions, which together with (3.38) and (3.39), form the level one super self-duality equations for $\lambda = -3$,

$$F^{(T_1^+)}\left(\underline{t}^+_{11}\right) = F^{(T_2^+)}\left(\underline{t}^+_{12}\right) = F^{(T_2^+)}\left(\underline{t}^+_{21}\right) = 0\,, \qquad F^{(T_1^+)}\left(\underline{t}^+_{22}\right) \neq 0\,. \tag{3.48}$$

The latter component (a singlet) clearly provides a source for the 7-dimensional part of $F^{(A)}$ required to be non zero. Again, as for $\lambda = 1$, there are no level two wet sources.

3.2 $H = \mathrm{Sp}(2) \otimes \mathrm{Sp}(1)/\mathbb{Z}_2 \subset \mathrm{Spin}(8)$

The decompositions of the representations in (3.26) to irreducible H-representations are tabulated below using labels $(\mathsf{ab,c})_{\underline{\mathbf{d}}}$, when (ab) and (c) are the Dynkin indices for Sp(2) and Sp(1) representations, respectively, and $\underline{\mathbf{d}}$ is the overall dimension.

X	$\rho_H(X)$
$W_3 = (0011)_{\mathbf{56_V}}$	$\{\underline{w}_{31} = (11,1)_{\underline{\mathbf{32}}}\,,\ \underline{w}_{32} = (10,3)_{\underline{\mathbf{16}}}\,,\ \underline{w}_{33} = (10,1)_{\underline{\mathbf{8}}}\}$
$W_1 = V = (1000)_{\mathbf{8}}$	$\{\underline{w}_1 = \underline{v} = (10,1)_{\underline{\mathbf{8}}}\}$
$U_4^+ = (0002)_{\mathbf{35_{S+}}}$	$\{\underline{u}^+_{41} = (02,0)_{\underline{\mathbf{14}}}\,,\ \underline{u}^+_{42} = (01,2)_{\underline{\mathbf{15}}}\,,\ \underline{u}^+_{43} = (00,4)_{\underline{\mathbf{5}}}\,,\ \underline{u}^+_{44} = (00,0)_{\underline{\mathbf{1}}}\}$
$U_0^+ = (0000)_{\mathbf{1}}$	$\{\underline{u}^+_0 = (00,0)_{\underline{\mathbf{1}}}\}$
$U_4^- = (0020)_{\mathbf{35_{S-}}}$	$\{\underline{u}^-_{41} = (20,2)_{\underline{\mathbf{30}}}\,,\ \underline{u}^-_{42} = (01,0)_{\underline{\mathbf{5}}}\}$
$U_0^- = (0000)_{\mathbf{1}}$	$\{\underline{u}^-_0 = (00,0)_{\underline{\mathbf{1}}}\}$
$T_1^+ = (1010)_{\mathbf{56_{S+}}}$	$\{\underline{t}^+_{11} = (20,2)_{\underline{\mathbf{30}}}\,,\ \underline{t}^+_{12} = (01,2)_{\underline{\mathbf{15}}}\,,\ \underline{t}^+_{13} = (20,0)_{\underline{\mathbf{10}}}\,,\ \underline{t}^+_{14} = (00,0)_{\underline{\mathbf{1}}}\}$
$T_2^+ = S^+ = (0001)_{\mathbf{8_{S+}}}$	$\{\underline{t}^+_{21} = \underline{s}^+_1 = (01,0)_{\underline{\mathbf{5}}}\,,\ \underline{t}^+_{22} = \underline{s}^+_2 = (00,2)_{\underline{\mathbf{3}}}\}$
$T_1^- = (1001)_{\mathbf{56_{S-}}}$	$\{\underline{t}^-_{11} = (11,1)_{\underline{\mathbf{32}}}\,,\ \underline{t}^-_{12} = (10,3)_{\underline{\mathbf{18}}}\,,\ \underline{t}^-_{13} = (10,1)_{\underline{\mathbf{8}}}\}$
$T_2^- = S^- = (0010)_{\mathbf{8_{S-}}}$	$\{\underline{t}^-_2 = \underline{s}^- = (10,1)_{\underline{\mathbf{8}}}\}$
$A = (0100)_{\mathbf{28}}$	$\{\underline{a}_1 = (20,0)_{\underline{\mathbf{10}}}\,,\ \underline{a}_2 = (01,2)_{\underline{\mathbf{15}}}\,,\ \underline{a}_3 = (00,2)_{\underline{\mathbf{3}}}\}$

The self-duality equations for this stability group were discussed in [3]. The three eigenvalues of the invariant T-tensor are $\lambda_{\underline{\mathbf{10}}} = 1$, $\lambda_{\underline{\mathbf{15}}} = -7/15$ and $\lambda_{\underline{\mathbf{3}}} = -1$, corresponding, respectively, to the eigenspaces $\underline{a}_1, \underline{a}_2$ and $\underline{a}_3$ into which A splits. Since the 8-dimensional euclidean tangent vectors transform as V, we may choose their basis in the form $X_{a\alpha}$, where $a = 1, \ldots, 4$ is an Sp(2) spinor index and $\alpha = 1, 2$ is an Sp(1) spinor index. Using the skew invariants C_{ab} and $\epsilon_{\alpha\beta}$ of Sp(2) and Sp(1), respectively, we obtain the decomposition of the vector-vector curvature tensor into the three irreducible descendants of the adjoint representation A:

$$\left[D^{(V)}_{a\alpha}, D^{(V)}_{b\beta}\right] = \epsilon_{\alpha\beta} F_{ab} + G_{ab\alpha\beta} + C_{ab} H_{\alpha\beta}\,, \tag{3.49}$$

where $F_{ab} = F_{ba}$ transforms as $\underline{a}_1$, $G_{ab\alpha\beta} = G_{ab\beta\alpha} = -G_{ba\alpha\beta}$ is "traceless", $C^{ab} G_{ab\alpha\beta} = 0$, and represents the $\underline{a}_2$ eigenspace; and $H_{\alpha\beta} = H_{\beta\alpha}$ transforms as $\underline{a}_3$.

To obtain the relevant source and sink subsets, we need the following Sp(2) tensor products:

$$\begin{aligned}
(01)_{\underline{\mathbf{5}}} \otimes (20)_{\underline{\mathbf{10}}} &= (21)_{\underline{\mathbf{35}}} \oplus (20)_{\underline{\mathbf{10}}} \oplus (01)_{\underline{\mathbf{5}}}\,,\\
(01)_{\underline{\mathbf{5}}} \otimes (11)_{\underline{\mathbf{16}}} &= (12)_{\underline{\mathbf{40}}} \oplus (30)_{\underline{\mathbf{20}}} \oplus (11)_{\underline{\mathbf{16}}} \oplus (10)_{\underline{\mathbf{4}}}\,,\\
(01)_{\underline{\mathbf{5}}} \otimes (10)_{\underline{\mathbf{4}}} &= (11)_{\underline{\mathbf{16}}} \oplus (10)_{\underline{\mathbf{4}}}\,,\\
(01)_{\underline{\mathbf{5}}} \otimes (01)_{\underline{\mathbf{5}}} &= (02)_{\underline{\mathbf{14}}} \oplus (20)_{\underline{\mathbf{10}}} \oplus (00)_{\underline{\mathbf{1}}}\,,\\
(10)_{\underline{\mathbf{4}}} \otimes (02)_{\underline{\mathbf{14}}} &= (12)_{\underline{\mathbf{40}}} \oplus (11)_{\underline{\mathbf{16}}}\,,\\
(10)_{\underline{\mathbf{4}}} \otimes (11)_{\underline{\mathbf{16}}} &= (21)_{\underline{\mathbf{35}}} \oplus (02)_{\underline{\mathbf{14}}} \oplus (20)_{\underline{\mathbf{10}}} \oplus (01)_{\underline{\mathbf{5}}}\,,\\
(10)_{\underline{\mathbf{4}}} \otimes (10)_{\underline{\mathbf{4}}} &= (20)_{\underline{\mathbf{10}}} \oplus (01)_{\underline{\mathbf{5}}} \oplus (00)_{\underline{\mathbf{1}}}\,.
\end{aligned} \tag{3.50}$$

Using these, we see that for all eigenvalues λ, we have

$$\begin{aligned}
\overset{\triangledown}{\sigma}_{H,\lambda}(V,V) &= \rho_H(V) = \{\underline{v}\}\,,\\
\overset{\triangledown}{\sigma}_{H,\lambda}\left(S^+, T^+_2\right) &= \rho_H\left(T^+_2\right),\\
\overset{\triangledown}{\sigma}_{H,\lambda}\left(S^-, T^-_i\right) &= \rho_H\left(T^-_i\right), \qquad i = 1, 2\,.
\end{aligned} \tag{3.51}$$

$\boldsymbol{\lambda_{\underline{\mathbf{10}}} = 1.}$ For this eigenvalue, the self-duality equations are [3]:

$$F^{(A)}(\underline{a}_2) = F^{(A)}(\underline{a}_3) = 0 \Longleftrightarrow G_{ab\alpha\beta} = H_{\alpha\beta} = 0\,, \tag{3.52}$$

with $F_{ab} \neq 0$. These equations may be written in the form,

$$\left[D^{(V)}_{a\alpha}, D^{(V)}_{b\beta}\right] = \epsilon_{\alpha\beta} F_{ab}\,. \tag{3.53}$$

They are particularly interesting, because they are in some sense solvable [3]. The level one sinks and wet sources are given by (3.51) together with

$$\overset{\triangledown}{\sigma}_{H,\lambda=1}\left(S^+, T^+_1\right) = \left\{\underline{t}^+_{11}, \underline{t}^+_{12}, \underline{t}^+_{14}\right\}, \tag{3.54}$$

$$\overset{\smile}{\sigma}_{H,\lambda=1}\left(S^+, T^+_1\right) = \left\{\underline{t}^+_{13}\right\}. \tag{3.55}$$

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Again, putting the curvatures corresponding to the sinks (3.51) and (3.54) to zero yields level one super self-duality equations for this eigenvalue, with the only non-zero supercurvature components given by,

$$\left[D^{(S^-)}_{a\alpha}, D^{(V)}_{b\beta}\right] = \epsilon_{\alpha\beta} f_{ab} \Longrightarrow \left[D^{(V)}_{a\alpha}, D^{(V)}_{b\beta}\right] = \epsilon_{\alpha\beta} F_{ab}\,, \tag{3.56}$$

where $f_{ab} = f_{ba}$ transforms as $\underline{t}^+_{13}$. At level two, there are no non-empty wet sources.

We note that the decompositions for the 8-dimensional $S^\pm$ do not contain spinors of Sp(2) or Sp(1), so our level one super self-dualities do not correspond to those suggested in [2] as supersymmetrisations of (3.53), namely,

$$\begin{aligned} [D_{a\alpha}, D_{b\beta}] &= \epsilon_{\alpha\beta} F_{ab}\,, \qquad [D_{a\alpha}, D_{\beta}] = \epsilon_{\alpha\beta} F_a\,, \qquad \{D_\alpha, D_\beta\} = 0\,, \\ \{D_a, D_\alpha\} &= D_{a\alpha}\,, \qquad \{D_a, D_b\} = 0\,, \qquad [D_a, D_{b\beta}] = 0\,, \end{aligned} \tag{3.57}$$

where the super covariant derivative $(D_{a\alpha}, D_\alpha, D_a)$, contains the Sp(1) and Sp(n) spinors D_α and D_a. The odd-odd and even-odd parts of these relations, which do not have a Spin(d)-origin, also lead to (3.53) in virtue of super Jacobi identities. (Similar lower level constraints implying the restriction to the other two eigenspaces may easily be determined).

$\boldsymbol{\lambda_{\underline{15}} = -7/15.}$ For this eigenvalue, the self-duality equations take the form:

$$F^{(A)}\left(\underline{a}_1\right) = F^{(A)}\left(\underline{a}_3\right) = 0 \Longleftrightarrow F_{ab} = H_{\alpha\beta} = 0\,, \tag{3.58}$$

in other words, $G_{ab\alpha\beta} \neq 0$. Here, we have

$$\overset{\triangledown}{\sigma}_{H,\lambda=-7/15}\left(S^+, T^+_1\right) = \rho_H\left(T^+_1\right), \tag{3.59}$$

in addition to (3.51). So there are no non-empty wet sources. However, two types of super self-duality equations may be considered:

(A) The non-maximal replacements, with

$$F^{(A)}(\underline{a}_3) = 0\,, \qquad F^{(T^+)}\left(\underline{t}^+\right) = 0\,, \qquad \text{for } \underline{t}^+ = \underline{t}^+_{11}, \underline{t}^+_{13}, \underline{t}^+_{14}, \underline{t}^+_{21} \tag{3.60}$$

imply (3.58). This leaves $F^{(T^+)}(\underline{t}^+_{12})$ and $F^{(T^+)}(\underline{t}^+_{22})$, which do not contribute to $F^{(A)}(\underline{a}_1)$, as non-vanishing sources for $F^{(A)}(\underline{a}_2)$.

(B) Alternatively, a non-zero $F^{(A)}(\underline{a}_2)$ may be obtained if any of the following consequences of (3.59),

$$F^{(T^+)}\left(\underline{t}^+_{11}\right) = F^{(T^+)}\left(\underline{t}^+_{12}\right) = F^{(T^+)}\left(\underline{t}^+_{21}\right) = F^{(T^+)}\left(\underline{t}^+_{22}\right) \tag{3.61}$$

are replaced by the respective chirality conditions

$$\begin{aligned} \left[D^{(S^+)}\left(\underline{s}^+_2\right), F^{(T^+)}\left(\underline{t}^+_{11}\right)\right] &= 0\,, \qquad \left[D^{(S^+)}\left(\underline{s}^+_1\right), F^{(T^+)}\left(\underline{t}^+_{12}\right)\right] = 0\,, \\ \left[D^{(S^+)}\left(\underline{s}^+_1\right), F^{(T^+)}\left(\underline{t}^+_{21}\right)\right] &= 0\,, \qquad \left[D^{(S^+)}\left(\underline{s}^+_2\right), F^{(T^+)}\left(\underline{t}^+_{22}\right)\right] = 0\,. \end{aligned} \tag{3.62}$$

If any of these higher order systems are used, their consistency conditions need to be checked. We note that the latter are systems of mixed order. The required self-duality equations arise as a consequence of a combination of linear relations among some curvature components, which are first order equations for the potentials, and first order equations for other curvature components, which are second order equations for the potentials.

$\lambda_{\underline{3}} = -1.$ For this eigenvalue, we have the equations

$$F^{(A)}(\underline{a}_1) = F^{(A)}(\underline{a}_2) = 0 \Longleftrightarrow F_{ab} = G_{ab\alpha\beta} = 0\,, \tag{3.63}$$

in other words, $H_{\alpha\beta} \neq 0$. Here, the level one sinks and wet sources are given by (3.51) together with

$$\begin{aligned} \overset{\triangledown}{\sigma}_{H,\lambda=-1}\left(S^+, T_1^+\right) &= \{\underline{t}_{11}^+, \underline{t}_{12}^+, \underline{t}_{13}^+\}\,, \\ \breve{\sigma}_{H,\lambda=-1}\left(S^+, T_1^+\right) &= \{\underline{t}_{14}^+\} \end{aligned} \tag{3.64}$$

yielding corresponding level one super self-duality equations, with non-zero supercurvature components given by

$$\left[D_{a\alpha}^{(S^-)}, D_{b\beta}^{(V)}\right] = C_{ab}\epsilon_{\alpha\beta}h \Longrightarrow \left[D_{a\alpha}^{(V)}, D_{b\beta}^{(V)}\right] = C_{ab}H_{\alpha\beta}\,, \tag{3.65}$$

where h corresponds to the singlet $\underline{t}_{14}^+$. At level two, there are no non-empty wet sources.

3.3 $H = \mathrm{SU}(2) \otimes \mathrm{SU}(2)/\mathbb{Z}_2 \subset \mathrm{Spin}(8)$

The calculation of the eigenvalues for SO(4)-invariant T-tensors is discussed in appendix B. In this case, we have three eigenspaces, $\underline{a}_1$, $\underline{a}_2 \oplus \underline{a}_3$ and $\underline{a}_4$ (see table below) with eigenvalues $\lambda_{\underline{15}} = 1$, $\lambda_{\underline{10}} = -3$ and $\lambda_{\underline{3}} = 5$, respectively. The decompositions of the representations in (3.26) to irreducible $H = \mathrm{SU}(2) \otimes \mathrm{SU}(2)/\mathbb{Z}_2$ representations are tabulated below using labels $(\mathsf{a},\mathsf{b})_{\underline{\mathbf{d}}}$, when (a) and (b) are the Dynkin indices for the two SU(2)'s and $\underline{\mathbf{d}}$ is the overall dimension.

X	$\rho_H(X)$
$W_3 = (0011)_{\mathbf{56_V}}$	$\{\underline{w}_{31} = (1,9)_{\underline{\mathbf{20}}},\ \underline{w}_{32} = (1,7)_{\underline{\mathbf{16}}},\ \underline{w}_{33} = (1,5)_{\underline{\mathbf{12}}},\ \underline{w}_{32} = (1,3)_{\underline{\mathbf{8}}}\}$
$W_1 = V = (1000)_{\mathbf{8}}$	$\{\underline{w}_1 = \underline{v} = (1,3)_{\underline{\mathbf{8}}}\}$
$U_4^+ = (0002)_{\mathbf{35_{S^+}}}$	$\{\underline{u}_{41}^+ = (0,12)_{\underline{\mathbf{13}}},\ \underline{u}_{42}^+ = (0,8)_{\underline{\mathbf{9}}},\ \underline{u}_{43}^+ = (0,6)_{\underline{\mathbf{7}}},$ $\underline{u}_{44}^+ = (0,4)_{\underline{\mathbf{5}}},\ \underline{u}_{45}^+ = (0,0)_{\underline{\mathbf{1}}}\}$
$U_0^+ = (0000)_{\mathbf{1}}$	$\{\underline{u}_0^+ = (0,0)_{\underline{\mathbf{1}}}\}$
$U_4^- = (0020)_{\mathbf{35_{S^-}}}$	$\{\underline{u}_{41}^- = (2,6)_{\underline{\mathbf{21}}},\ \underline{u}_{42}^- = (2,2)_{\underline{\mathbf{9}}},\ \underline{u}_{43}^- = (0,4)_{\underline{\mathbf{5}}}\}$
$U_0^- = (0000)_{\mathbf{1}}$	$\{\underline{u}_0^- = (0,0)_{\underline{\mathbf{1}}}\}$
$T_1^+ = (1010)_{\mathbf{56_{S^+}}}$	$\{\underline{t}_{11}^+ = (2,6)_{\underline{\mathbf{21}}},\ \underline{t}_{12}^+ = (2,4)_{\underline{\mathbf{15}}},\ \underline{t}_{13}^+ = (2,2)_{\underline{\mathbf{9}}},$ $\underline{t}_{14}^+ = (0,4)_{\underline{\mathbf{5}}},\ \underline{t}_{15}^+ = (2,0)_{\underline{\mathbf{3}}},\ \underline{t}_{16}^+ = (0,2)_{\underline{\mathbf{3}}}\}$
$T_2^+ = S^+ = (0001)_{\mathbf{8_{S^+}}}$	$\{\underline{t}_{21}^+ = \underline{s}_1^+ = (0,6)_{\underline{\mathbf{7}}},\ \underline{t}_{22}^+ = \underline{s}_2^+ = (0,0)_{\underline{\mathbf{1}}}\}$
$T_1^- = (1001)_{\mathbf{56_{S^-}}}$	$\{\underline{t}_{11}^- = (1,9)_{\underline{\mathbf{20}}},\ \underline{t}_{12}^- = (1,7)_{\underline{\mathbf{16}}},\ \underline{t}_{13}^- = (1,5)_{\underline{\mathbf{12}}},\ \underline{t}_{14}^- = (1,3)_{\underline{\mathbf{8}}}\}$
$T_2^- = S^- = (0010)_{\mathbf{8_{S^-}}}$	$\{\underline{t}_2^- = \underline{s}^- = (1,3)_{\underline{\mathbf{8}}}\}$
$A = (0100)_{\mathbf{28}}$	$\{\underline{a}_1 = (2,4)_{\underline{\mathbf{15}}},\ \underline{a}_2 = (0,6)_{\underline{\mathbf{7}}},\ \underline{a}_3 = (0,2)_{\underline{\mathbf{3}}},\ \underline{a}_4 = (2,0)_{\underline{\mathbf{3}}}\}$

Writing tangent vectors in 2-spinor notation, $X_{\alpha\dot\alpha\dot\beta\dot\gamma}$ (completely symmetric in the dotted indices), the decomposition of the curvature tensor into the four irreducible descendants of the adjoint representation A may be expressed as,

$$\begin{aligned}\left[D^{(V)}_{\alpha\dot\alpha_1\dot\alpha_2\dot\alpha_3}, D^{(V)}_{\beta\dot\beta_1\dot\beta_2\dot\beta_3}\right] = \epsilon_{(\dot\alpha_1\dot\beta_1}\ F_{\dot\alpha_2\dot\alpha_3\dot\beta_2\dot\beta_3)\alpha\beta} + \epsilon_{\alpha\beta}\ G_{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\beta_1\dot\beta_2\dot\beta_3} +& \\ + \epsilon_{\alpha\beta}\epsilon_{(\dot\alpha_1\dot\beta_1}\epsilon_{\dot\alpha_2\dot\beta_2}\ G_{\dot\alpha_3\dot\beta_3)} + \epsilon_{(\dot\alpha_1\dot\beta_1}\epsilon_{\dot\alpha_2\dot\beta_2}\epsilon_{\dot\alpha_3\dot\beta_3)}\ H_{\alpha\beta}\,,& \qquad (3.66)\end{aligned}$$

where the brackets () around indices denote symmetrisation and the curvature tensors are separately symmetric under interchange of dotted and undotted indices. Thus, the tensors $F_{\alpha\beta\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}$, $G_{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4\dot\alpha_5\dot\alpha_6}$, $G_{\dot\alpha\dot\beta}$ and $H_{\alpha\beta}$ are irreducible under $\mathrm{SU}(2)\otimes\mathrm{SU}(2)$, transforming under the representations $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ and $\underline{a}_4$, respectively. The T-tensor corresponds to the singlet $\underline{u}^+_{45}$ and the self-duality conditions take the form of the vanishing of certain irreducible parts of the curvature [4]. Curvature constraints, which occur as integrability conditions for certain covariant-constancy conditions, have also been considered [3]. The latter, however, do not correspond to eigenvalue conditions for a T-tensor (and hence do not imply the Yang-Mills equations). We also note that the descendants of S^+ and S^- do not contain spinor representations of the subgroup H, so the systems of super selfduality equations sought here do not correspond to those considered in [4].

$\boldsymbol{\lambda_{\underline{15}} = 1.}$ For this eigenvalue, we obtain the equations

$$F^{(A)}(\underline{a}_2) = F^{(A)}(\underline{a}_3) = F^{(A)}(\underline{a}_4) = 0 \Longleftrightarrow G_{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4\dot\alpha_5\dot\alpha_6} = G_{\dot\alpha\dot\beta} = H_{\alpha\beta} = 0\,, \qquad (3.67)$$

with $F_{\alpha\beta\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4} \neq 0$. We find the corresponding sink and wet source subsets to be,

$$\begin{aligned}\overset{\triangledown}{\sigma}_{H,\lambda=1}(V,V) &= \rho_H(V) = \{\underline{v}\}\,,\\ \overset{\triangledown}{\sigma}_{H,\lambda=1}\left(S^+,T_1^+\right) &= \{\underline{t}^+_{11},\underline{t}^+_{14},\underline{t}^+_{15},\underline{t}^+_{16}\}\,,\\ \overset{\triangledown}{\sigma}_{H,\lambda=1}\left(S^+,T_2^+\right) &= \rho_H\left(T_2^+\right)\,,\\ \overset{\triangledown}{\sigma}_{H,\lambda=1}\left(S^-,T_i^-\right) &= \rho_H\left(T_i^-\right)\,,\\ \breve{\sigma}_{H,\lambda=1}\left(S^+,T_1^+\right) &= \{\underline{t}^+_{12},\underline{t}^+_{13}\}\,. \end{aligned}\qquad (3.68)$$

These yield level one super self-duality equations, with non-zero supercurvature components $f_{\dot\alpha\dot\beta\dot\gamma\dot\delta\alpha\beta}$ and $f_{\dot\alpha\dot\beta\alpha\beta}$ transforming as $\underline{t}^+_{12}$ and $\underline{t}^+_{13}$, respectively, and given by,

$$\left[D^{(S^-)}_{\alpha\dot\alpha_1\dot\alpha_2\dot\alpha_3}, D^{(V)}_{\beta\dot\beta_1\dot\beta_2\dot\beta_3}\right] = \epsilon_{(\dot\alpha_1\dot\beta_1}\ f_{\dot\alpha_2\dot\alpha_3\dot\beta_2\dot\beta_3)\alpha\beta} + \epsilon_{(\dot\alpha_1\dot\beta_1}\epsilon_{\dot\alpha_2\dot\beta_2}\ f_{\dot\alpha_3\dot\beta_3)\alpha\beta}\,. \qquad (3.69)$$

In virtue of the super Jacobi identities, these imply the level zero self-duality equations,

$$\left[D^{(V)}_{\alpha\dot\alpha_1\dot\alpha_2\dot\alpha_3}, D^{(V)}_{\beta\dot\beta_1\dot\beta_2\dot\beta_3}\right] = \epsilon_{(\dot\alpha_1\dot\beta_1}\ F_{\dot\alpha_2\dot\alpha_3\dot\beta_2\dot\beta_3)\alpha\beta}\,. \qquad (3.70)$$

At level two, there are no non-empty wet sources.

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$\boldsymbol{\lambda_{\underline{10}} = -3.}$ For this eigenvalue, we have the equations

$$F^{(A)}(\underline{a}_1) = F^{(A)}(\underline{a}_4) = 0 \Longleftrightarrow F_{\alpha\beta\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4} = H_{\alpha\beta} = 0\,, \tag{3.71}$$

with $G_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4\dot{\alpha}_5\dot{\alpha}_6} \neq 0$, $G_{\dot{\alpha}\dot{\beta}} \neq 0$. These are implied by level one super self-duality equations corresponding to the sink and wet source subsets,

$$\begin{aligned}
\overset{\triangledown}{\sigma}_{H,\lambda=-3}(V,V) &= \rho_H(V) = \{\underline{v}\}\,,\\
\overset{\triangledown}{\sigma}_{H,\lambda=-3}\left(S^+,T_1^+\right) &= \left\{\underline{t}_{11}^+,\underline{t}_{12}^+,\underline{t}_{13}^+,\underline{t}_{15}^+\right\}\,,\\
\overset{\triangledown}{\sigma}_{H,\lambda=-3}\left(S^-,T_1^-\right) &= \left\{\underline{t}_{12}^-,\underline{t}_{13}^-,\underline{t}_{14}^-\right\}\,,\\
\overset{\triangledown}{\sigma}_{H,\lambda=-3}\left(S^-,T_2^-\right) &= \rho_H\left(T_2^-\right)\,,\\
\breve{\sigma}_{H,\lambda=-3}\left(S^+,T_1^+\right) &= \left\{\underline{t}_{14}^+,\underline{t}_{16}^+\right\}\,,\\
\breve{\sigma}_{H,\lambda=-3}\left(S^+,T_2^+\right) &= \rho_H\left(T_2^+\right)\,,\\
\breve{\sigma}_{H,\lambda=-3}\left(S^-,T_1^-\right) &= \left\{\underline{t}_{11}^-\right\}\,.
\end{aligned} \tag{3.72}$$

At level two, non-trivial equations are obtained corresponding to the sinks and wet sources,

$$\begin{aligned}
\overset{\triangledown}{\tau}_{H,\lambda=-3}\left(S^\pm,T_i^\pm;S^\mp,W\right) &= \rho_H(W)\,,\\
\overset{\triangledown}{\tau}_{H,\lambda=-3}\left(S^+,T_1^+;S^+,U^-\right) &= \left\{\underline{u}_{41}^-,\underline{u}_{42}^-\right\}\,,\\
\breve{\tau}_{H,\lambda=-3}\left(S^+,T_1^+;S^+,U^-\right) &= \left\{\underline{u}_{43}^-\right\}\,,\\
\breve{\tau}_{H,\lambda=-3}\left(S^+,T_2^+;S^+,U^-\right) &= \left\{\underline{u}_{43}^-,\underline{u}_0^-\right\}\,.
\end{aligned} \tag{3.73}$$

Thus, the only non-zero level two supercurvatures are those transforming as $\underline{u}_{43}^-,\underline{u}_0^-$:

$$\left\{D^{(S^-)}_{\alpha\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3}, D^{(S^-)}_{\beta\dot{\beta}_1\dot{\beta}_2\dot{\beta}_3}\right\} = \epsilon_{\alpha\beta}\;\epsilon_{\dot{\alpha}_1\dot{\beta}_1}\;\gamma_{\dot{\alpha}_2\dot{\alpha}_3\dot{\beta}_2\dot{\beta}_3} + \epsilon_{\alpha\beta}\epsilon_{(\dot{\alpha}_1\dot{\beta}_1}\epsilon_{\dot{\alpha}_2\dot{\beta}_2}\;\epsilon_{\dot{\alpha}_3\dot{\beta}_3)}\gamma\,. \tag{3.74}$$

These imply that at level one, the only non-zero supercurvatures are those given by,

$$\begin{aligned}
\left[D^{(S^-)}_{\alpha\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3}, D^{(V)}_{\beta\dot{\beta}_1\dot{\beta}_2\dot{\beta}_3}\right] &= \epsilon_{\alpha\beta}\,\epsilon_{\dot{\alpha}_1\dot{\beta}_1}\;g_{\dot{\alpha}_2\dot{\alpha}_3\dot{\beta}_2\dot{\beta}_3} + \epsilon_{\alpha\beta}\,\epsilon_{(\dot{\alpha}_1\dot{\beta}_1}\epsilon_{\dot{\alpha}_2\dot{\beta}_2}\,g_{\dot{\alpha}_3\dot{\beta}_3)} +\\
&\quad + \epsilon_{\alpha\beta}\;g_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\beta}_1\dot{\beta}_2\dot{\beta}_3} + \epsilon_{\alpha\beta}\epsilon_{(\dot{\alpha}_1\dot{\beta}_1}\epsilon_{\dot{\alpha}_2\dot{\beta}_2}\;\epsilon_{\dot{\alpha}_3\dot{\beta}_3)}\,g\,,\\
\left[D^{(S^+)}_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4\dot{\alpha}_5\dot{\alpha}_6}, D^{(V)}_{\beta\dot{\beta}_1\dot{\beta}_2\dot{\beta}_3}\right] &= g_{\beta\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4\dot{\alpha}_5\dot{\alpha}_6\dot{\beta}_1\dot{\beta}_2\dot{\beta}_3}\,,
\end{aligned} \tag{3.75}$$

with supercurvature components transforming as $\underline{t}_{14}^+,\underline{t}_{16}^+,\underline{t}_{21}^+,\underline{t}_{22}^+$ and $\underline{t}_{11}^-$. In turn, these imply the level zero constraints (3.71), i.e.

$$\left[D^{(V)}_{\alpha\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3}, D^{(V)}_{\beta\dot{\beta}_1\dot{\beta}_2\dot{\beta}_3}\right] = \epsilon_{\alpha\beta}\;G_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\beta}_1\dot{\beta}_2\dot{\beta}_3} + \epsilon_{\alpha\beta}\epsilon_{(\dot{\alpha}_1\dot{\beta}_1}\epsilon_{\dot{\alpha}_2\dot{\beta}_2}\;G_{\dot{\alpha}_3\dot{\beta}_3)}\,. \tag{3.76}$$

$\boldsymbol{\lambda_{\underline{3}} = 5.}$ For this eigenvalue, we have the equations

$$\begin{aligned}
&F^{(A)}(\underline{a}_1) = F^{(A)}(\underline{a}_2) = F^{(A)}(\underline{a}_3) = 0\\
\Longleftrightarrow\;\; &F_{\alpha\beta\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4} = G_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4\dot{\alpha}_5\dot{\alpha}_6} = G_{\dot{\alpha}\dot{\beta}} = 0\,,
\end{aligned} \tag{3.77}$$

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with $H_{\alpha\beta} \neq 0$. These are implied by level one super self-duality equations corresponding to the sink and wet source subsets,

$$\begin{aligned}
\overset{\triangledown}{\sigma}_{H,\lambda=5}(V,V) &= \rho_H(V) = \{\underline{v}\}\,, \\
\overset{\triangledown}{\sigma}_{H,\lambda=5}\left(S^+,T_1^+\right) &= \{\underline{t}_{11}^+,\underline{t}_{12}^+,\underline{t}_{13}^+,\underline{t}_{14}^+,\underline{t}_{16}^+\}\,, \\
\overset{\triangledown}{\sigma}_{H,\lambda=5}\left(S^+,T_2^+\right) &= \rho_H\left(T_2^+\right), \\
\overset{\triangledown}{\sigma}_{H,\lambda=5}\left(S^-,T_i^-\right) &= \rho_H\left(T_i^-\right), \\
\breve{\sigma}_{H,\lambda=5}\left(S^+,T_1^+\right) &= \{\underline{t}_{15}^+\}\,.
\end{aligned} \tag{3.78}$$

There are no non-trivial level two conditions for this eigenvalue. Thus, the only non-zero supercurvatures are those corresponding to $\underline{t}_{15}^+$ and $\underline{a}_4$ in:

$$\begin{aligned}
&\left[D^{(S^-)}_{\alpha\dot\alpha_1\dot\alpha_2\dot\alpha_3}, D^{(V)}_{\beta\dot\beta_1\dot\beta_2\dot\beta_3}\right] = \epsilon_{(\dot\alpha_1\dot\beta_1}\,\epsilon_{\dot\alpha_2\dot\beta_2}\epsilon_{\dot\alpha_3\dot\beta_3)}\;h_{\alpha\beta} \\
\Longrightarrow\;&\left[D^{(V)}_{\alpha\dot\alpha_1\dot\alpha_2\dot\alpha_3}, D^{(V)}_{\beta\dot\beta_1\dot\beta_2\dot\beta_3}\right] = \epsilon_{(\dot\alpha_1\dot\beta_1}\,\epsilon_{\dot\alpha_2\dot\beta_2}\epsilon_{\dot\alpha_3\dot\beta_3)}H_{\alpha\beta}\,.
\end{aligned} \tag{3.79}$$

4. Case of $d = 5, 6, 7 \pmod 8$

For these dimensions, the vector appears in the skewsymmetric square of any fundamental spinor representation $S \in \Sigma$. We therefore need at least two copies of the same spinor representation S, i.e. $N = 2$ is the "minimal" model. Spin(d) for d odd (here $d = 5, 7 \pmod 8$) has only one fundamental spinor representation S. However, Spin(d) for $d = 6 \pmod 8$ has two spinor representations S^+ and S^-, with the vector arising in both $S^+ \otimes S^+$ and $S^- \otimes S^-$. We will only consider the *chiral* superspace, in which the S^- representation does not act and we denote S^+ by S. Our analysis affords straightforward extension to the non-chiral cases. The curvatures (with $i = 1, 2$) are defined by

$$\begin{aligned}
&\left\{D_A^{(S)i}, D_B^{(S)j}\right\} = \epsilon^{ij} C(S,A,S,B;V,M)\; D_M^{(V)} + \sum_{U\in\{S\otimes S\}} C(S,A,S,B;U,L)\; F_L^{(U)ij}\,, \\
&\left[D_A^{(S)i}, D_M^{(V)}\right] = \sum_{T\in\{S\otimes V\}} C(S,A,V,M;T,D)\; F_D^{(T)i}\,, \qquad \left[D_M^{(V)}, D_N^{(V)}\right] = F_{MN}^{(A)}\,.
\end{aligned} \tag{4.1}$$

Here, the Clebsch-Gordon coefficients $C(S,A,S,B;V,M)$ are antisymmetrical in A, B and $F^{(U)ij}$ is symmetric or antisymmetric in i, j for representations U occuring as summands in $\vee^2 S$ or $\wedge^2 S$, respectively. From (A.5), (A.7), (A.38) and (A.39), the U's are given by

$$\begin{aligned}
&U \in \left\{R(2\pi_r)\,,\; R(\pi_{2p+1})\,;\; 0 \le p \le \frac{(r-3)}{2}\right\}, \qquad \text{for } d = 6 \mod 8\,, \\
&U \in \{R(2\pi_r)\,,\; R(\pi_p)\,;\; 0 \le p \le r-1\}\,, \qquad \text{for } d = 5,7 \mod 8
\end{aligned} \tag{4.2}$$

and from (A.3) and (A.37) we see that

$$\begin{aligned}
T_1 &= R(\pi_1 + \pi_r)\,, \qquad T_2 = R(\pi_{r-1})\,, \qquad \text{for even } d\,, \\
T_1 &= R(\pi_1 + \pi_r)\,, \qquad T_2 = R(\pi_r)\,, \qquad \text{for odd } d\,.
\end{aligned} \tag{4.3}$$

Following the same pattern as in the previous case, we consider the first and second level Jacobi identities.

Level 1 super self-duality. The first level Jacobi involves $\{D_A^{(S)1}, D_B^{(S)2}, D_M^{(V)}\}$ and since $F_N^{(V)ij} = \epsilon^{ij} F_N^{(V)}$, it yields,

$$
\begin{aligned}
F_{NM}^{(A)} = {}& \frac{1}{2}\left(\left[D_N^{(V)}, F_M^{(V)}\right] - \left[D_M^{(V)}, F_N^{(V)}\right]\right) + \\
& + \sum_{T \in \{S \otimes V\}} \alpha_1(T)\, C(A, NM; T, D, S, C)\, \epsilon_{ij} \left\{D_C^{(S)i}, F_D^{(T)j}\right\}, \qquad (4.4)
\end{aligned}
$$

where, as before, α_1 incorporates recoupling coefficients. We see from (A.23), (A.28), (A.37) and (A.40), that for $d = 5, 6, 7 \pmod 8$, the adjoint $\wedge^2 V$ is always contained in the decomposition of $T_i \otimes S$ for $i = 1, 2$. Thus, in order to guarantee that $F^{(A)}$, under H-decomposition, is restricted to its components in $\rho_H(\lambda)$, it suffices to impose

$$
F^{(V)}(\underline{v}) = 0, \qquad \text{for all } \underline{v} \in \overset{\triangledown}{\sigma}_{H,\lambda}(V, V), \qquad (4.5)
$$

$$
F^{(T)i}(\underline{t}) = 0, \qquad \text{for all } \underline{t} \in \overset{\triangledown}{\sigma}_{H,\lambda}(S, T). \qquad (4.6)
$$

In order to have a non-trivial $F^{(A)}$ satisfying (1.1), we require, in addition, that after imposing (4.5) and (4.6) we still have

$$
F^{(A)}(\underline{a}) \neq 0, \qquad \text{for at least one } \underline{a} \in \rho_H(\lambda). \qquad (4.7)
$$

This is guaranteed if the following set of curvature components is non empty:

$$
\left\{F^{(V)}(\underline{v}), F^{(T)i}(\underline{t}) \;\middle|\; \underline{v} \in \breve{\sigma}_{H,\lambda}(V, V),\ \underline{t} \in \breve{\sigma}_{H,\lambda}(S, T)\right\}. \qquad (4.8)
$$

Level 2 super self-duality. The second level Jacobi identities are obtained from $\{D_A^{(S)i}$, $D_B^{(S)j}, D_C^{(S)k}\}$. They take the bare form

$$
\begin{aligned}
& C(S, A, S, B; V, M)\ C(S, C, V, M; T, D)\epsilon^{ij}\ F_D^{(T)k} + \\
& + C(S, B, S, C; V, M)\ C(S, A, V, M; T, D)\epsilon^{jk}\ F_D^{(T)i} + \\
& + C(S, C, S, A; V, M)\ C(S, B, V, M; T, D)\epsilon^{ki}\ F_D^{(T)j} = \\
& = - \sum_{U \in \{S \otimes S\}} \left\{C(S, A, S, B; U, L)\left[D_C^{(S)k}, F_L^{(U)ij}\right] + C(S, B, S, C; U, L) \times \right. \\
& \qquad \left. \times \left[D_A^{(S)i}, F_L^{(U)jk}\right] + C(S, C, S, A; U, L)\left[D_B^{(S)j}, F_L^{(U)ki}\right]\right\}. \qquad (4.9)
\end{aligned}
$$

Now, using properties of the Clebsch-Gordon coefficients and of ϵ^{ij}, every curvature component $F_D^{(T)i}$ can be separately extracted. The tensor product decompositions (A.32), (A.34), (A.44) and (A.45) show that $S \otimes U$ contain both T_1 and T_2 except for $R(2\pi_r) \otimes R(\pi_r)$ (for $d = 6 \pmod 8$), which does not yield T_2. We also note that $F^{(T)1}$ depends on $F^{(U)12}$ and $F^{(U)11}$, whereas $F^{(T)2}$ depends on $F^{(U)12}$ and $F^{(U)22}$.

Sufficient conditions for the satisfaction of (4.6) are,

$$
F^{(U)ij}(\underline{u}) = 0, \qquad \text{for all } \underline{u} \in \overset{\triangledown}{\tau}_{H,\lambda}(S, T; S, U). \qquad (4.10)
$$

The nontriviality condition is that the following set of superfields is non empty:

$$
\left\{F^{(U)ij}(\underline{u}) \;\middle|\; \underline{u} \in \breve{\tau}_{H,\lambda}(S, T; S, U)\right\}. \qquad (4.11)
$$

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4.1 $H = (\mathrm{SU}(3) \otimes \mathrm{U}(1))/\mathbb{Z}_3 \subset \mathrm{Spin}(6) = \mathrm{SU}(4)$

Using two copies of spinor representation $S = (001)_{\mathbf{4}}$ and vector $V = (100)_{\mathbf{6}}$, we have the Spin(6) representation spaces

$$
\begin{aligned}
U \in \{S \otimes S\} &= \{U_3 = (002)_{\mathbf{10}}\,, \qquad U_1 = V = (100)_{\mathbf{6}}\}\,,\\
T \in \{S \otimes V\} &= \{T_1 = (101)_{\mathbf{20}}\,, \qquad T_2 = (010)_{\mathbf{4}}\}\,,\\
A = \wedge^2 V &= (011)_{\mathbf{15}}\,,
\end{aligned}
\tag{4.12}
$$

which determine the Spin(6) covariant supercurvatures components. Under the breaking $\mathrm{Spin}(6) \supset (\mathrm{SU}(3) \otimes \mathrm{U}(1))/\mathbb{Z}_3$ the decompositions of the relevant representation spaces are tabulated below. We denote representations of the subgroup by $(\mathsf{ab})^c_{\underline{\mathbf{d}}}$, where (ab) are the Dynkin labels of SU(3), c is the U(1) eigenvalue and $\underline{d}$ is the dimension of the representation.

X	$\rho_H(X)$
$U_3 = (002)_{\mathbf{10}}$	$\{\underline{u}_{31} = (02)^{-2}_{\underline{\mathbf{6}}}\,,\ \underline{u}_{32} = (01)^{2}_{\underline{\mathbf{3}}}\,,\ \underline{u}_{33} = (00)^{6}_{\underline{\mathbf{1}}}\}$
$U_1 = V = (100)_{\mathbf{6}}$	$\{\underline{u}_{11} = \underline{v}_1 = (01)^{2}_{\underline{\mathbf{3}}}\,,\ \underline{u}_{12} = \underline{v}_2 = (10)^{-2}_{\underline{\mathbf{3}}}\}$
$T_1 = (101)_{\mathbf{20}}$	$\{\underline{t}_{11} = (11)^{-3}_{\underline{\mathbf{8}}}\,,\ \underline{t}_{12} = (02)^{1}_{\underline{\mathbf{6}}}\,,\ \underline{t}_{13} = (01)^{5}_{\underline{\mathbf{3}}}\,,\ \underline{t}_{14} = (10)^{1}_{\underline{\mathbf{3}}}\}$
$T_2 = (010)_{\mathbf{4}}$	$\{\underline{t}_{21} = (10)^{1}_{\underline{\mathbf{3}}}\,,\ \underline{t}_{22} = (00)^{-3}_{\underline{\mathbf{1}}}\}$
$S = (001)_{\mathbf{4}}$	$\{\underline{s}_1 = (01)^{-1}_{\underline{\mathbf{3}}}\,,\ \underline{s}_2 = (00)^{3}_{\underline{\mathbf{1}}}\}$
$A = (011)_{\mathbf{15}}$	$\{\underline{a}_1 = (11)^{0}_{\underline{\mathbf{8}}}\,,\ \underline{a}_2 = (10)^{4}_{\underline{\mathbf{3}}}\,,\ \underline{a}_3 = (01)^{-4}_{\underline{\mathbf{3}}}\,,\ \underline{a}_4 = (00)^{0}_{\underline{\mathbf{1}}}\}$.

The completely antisymmetric T_{MNPQ} tensor belongs to the adjoint representation (011) which contains the H singlet $\underline{a}_4$. The curvature $F^{(A)}$ decomposes into three eigenspaces, $\underline{a}_1, \underline{a}_2 \oplus \underline{a}_3$ and $\underline{a}_4$, having eigenvalues $\lambda_{\underline{\mathbf{8}}} = 1$, $\lambda_{\underline{\mathbf{6}}} = -1$ and $\lambda_{\underline{\mathbf{1}}} = -2$, respectively. The corresponding equations were explicitly displayed in [1]. For all eigenspaces, we have that $\overset{\triangledown}{\sigma}_{H,\lambda}(V,V) = \rho_H(V)$ and $\overset{\triangledown}{\sigma}_{H,\lambda}(S,T_2) = \rho_H(T_2)$, since the tensor products contributing to $S \otimes T_2$ are:

$$
\begin{aligned}
\left((01)^{-1}_{\underline{\mathbf{3}}} \oplus (00)^{3}_{\underline{\mathbf{1}}}\right) \otimes (10)^{1}_{\underline{\mathbf{3}}} &= (10)^{4}_{\underline{\mathbf{3}}} \oplus (11)^{0}_{\underline{\mathbf{8}}} \oplus (00)^{0}_{\underline{\mathbf{1}}}\,,\\
\left((01)^{-1}_{\underline{\mathbf{3}}} \oplus (00)^{3}_{\underline{\mathbf{1}}}\right) \otimes (00)^{-3}_{\underline{\mathbf{1}}} &= (01)^{-4}_{\underline{\mathbf{3}}} \oplus (00)^{0}_{\underline{\mathbf{1}}}\,.
\end{aligned}
\tag{4.13}
$$

The level one super selfduality systems therefore include, for any λ,

$$
F^{(V)}\left(\underline{v}_p\right) = F^{(T_2)i}\left(\underline{t}_{2p}\right) = 0\,, \qquad \text{for all } p\,. \tag{4.14}
$$

Imposing restrictions on various components of $F^{(T_1)}$ distinguishes the three self-dualities. The tensor products contributing to $S \otimes T_1$ are:

$$
\begin{aligned}
(\underline{s}_1 \oplus \underline{s}_2) \otimes \underline{t}_{11} &= \left((01)^{-1}_{\underline{\mathbf{3}}} \oplus (00)^{3}_{\underline{\mathbf{1}}}\right) \otimes (11)^{-3}_{\underline{\mathbf{8}}} = (11)^{0}_{\underline{\mathbf{8}}} \oplus (12)^{-4}_{\underline{\mathbf{15}}} \oplus (01)^{-4}_{\underline{\mathbf{3}}} \oplus (20)^{-4}_{\underline{\mathbf{6}}}\,,\\
(\underline{s}_1 \oplus \underline{s}_2) \otimes \underline{t}_{12} &= \left((01)^{-1}_{\underline{\mathbf{3}}} \oplus (00)^{3}_{\underline{\mathbf{1}}}\right) \otimes (02)^{1}_{\underline{\mathbf{6}}} = (02)^{4}_{\underline{\mathbf{6}}} \oplus (03)^{0}_{\underline{\mathbf{10}}} \oplus (11)^{0}_{\underline{\mathbf{8}}}\,,\\
(\underline{s}_1 \oplus \underline{s}_2) \otimes \underline{t}_{13} &= \left((01)^{-1}_{\underline{\mathbf{3}}} \oplus (00)^{3}_{\underline{\mathbf{1}}}\right) \otimes (01)^{5}_{\underline{\mathbf{3}}} = (01)^{8}_{\underline{\mathbf{3}}} \oplus (02)^{4}_{\underline{\mathbf{6}}} \oplus (10)^{4}_{\underline{\mathbf{3}}}\,,\\
(\underline{s}_1 \oplus \underline{s}_2) \otimes \underline{t}_{14} &= \left((01)^{-1}_{\underline{\mathbf{3}}} \oplus (00)^{3}_{\underline{\mathbf{1}}}\right) \otimes (10)^{1}_{\underline{\mathbf{3}}} = (10)^{4}_{\underline{\mathbf{3}}} \oplus (11)^{0}_{\underline{\mathbf{8}}} \oplus (00)^{0}_{\underline{\mathbf{1}}}\,.
\end{aligned}
\tag{4.15}
$$

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$\boldsymbol{\lambda_{\underline{8}} = 1.}$ The eigenspace $\rho_H(\lambda = 1) = \{\underline{a}_1 = (11)^0_{\underline{\mathbf{8}}}\}$ corresponds to 7 conditions on the 15 components of the curvature [1]:

$$F^{(A)}(\underline{a}_2) = F^{(A)}(\underline{a}_3) = F^{(A)}(\underline{a}_4) = 0\,. \tag{4.16}$$

From (4.15) we see that the wet source $\underline{t}_{12} = (02)^1_{\underline{\mathbf{6}}}$ yields a non-trivial contribution to $\underline{a}_1$. We therefore have

$$\overset{\triangledown}{\sigma}_{H,\lambda=1}(S,T_1) = \{\underline{t}_{11},\underline{t}_{13},\underline{t}_{14}\}\,, \qquad \text{and} \qquad \breve{\sigma}_{H,\lambda=1}(S,T_1) = \{\underline{t}_{12}\}\,, \tag{4.17}$$

yielding, in addition to (4.14), the level one super self-duality equations

$$F^{(T_2)i}(\underline{t}_{11}) = F^{(T_2)i}(\underline{t}_{13}) = F^{(T_2)i}(\underline{t}_{14}) = 0\,, \qquad F^{(T_2)i}(\underline{t}_{12}) \neq 0\,. \tag{4.18}$$

These equations are not implied by any non-trivial level two conditions.

$\boldsymbol{\lambda_{\underline{6}} = -1.}$ The eigenspace $\rho_H(\lambda = -1) = \{\underline{a}_2 \oplus \underline{a}_3 = (10)^4_{\underline{\mathbf{3}}} \oplus (01)^{-4}_{\underline{\mathbf{3}}}\}$ corresponds to 9 conditions on the 15 components of the curvature [1]:

$$F^{(A)}(\underline{a}_1) = F^{(A)}(\underline{a}_4) = 0\,. \tag{4.19}$$

Here we find

$$\overset{\triangledown}{\sigma}_{H,\lambda=-1}(S,T_1) = \{\underline{t}_{11},\underline{t}_{12},\underline{t}_{14}\}\,, \qquad \text{and} \qquad \breve{\sigma}_{H,\lambda=-1}(S,T_1) = \{\underline{t}_{13}\}\,, \tag{4.20}$$

yielding as level one super self-duality equations, together with (4.14),

$$F^{(T_2)i}(\underline{t}_{11}) = F^{(T_2)i}(\underline{t}_{12}) = F^{(T_2)i}(\underline{t}_{14}) = 0\,. \tag{4.21}$$

From (4.15) we see that the wet source $\underline{t}_{13} = (01)^5_{\underline{\mathbf{3}}}$ yields a non-trivial contribution to $\underline{a}_2$. Again, there are no non-trivial level two conditions.

$\boldsymbol{\lambda_{\underline{1}} = -2.}$ The self-duality conditions for this eigenvalue with eigenspace $\rho_H(\lambda = -2) = \{\underline{a}_4 = (00)^0_{\underline{\mathbf{1}}}\}$ correspond to the rather trivial set of 14 conditions on the 15 components of the curvature:

$$F^{(A)}(\underline{a}_1) = F^{(A)}(\underline{a}_2) = F^{(A)}(\underline{a}_3) = 0\,. \tag{4.22}$$

Here we have $\overset{\triangledown}{\sigma}_{H,\lambda=-1}(S,T_1) = \rho_H(T_1)$, so there are no wet sources and in order to have (4.22), we need to impose $F^{(T_1)i}(\underline{t}_{1j}) = 0$ for all j. One way to obtain (4.22) as consequences of the Bianchi identities, is to replace $F^{(T_2)i}(\underline{t}_{22}) = 0$ from (4.14) by the chirality condition

$$\Big[D^{(S)}(\underline{s}_1), F^{(T_2)i}(\underline{t}_{22})\Big] = 0\,. \tag{4.23}$$

Alternatively, we can take $F^{(T_2)i}(\underline{t}_{22}) \neq 0$ and impose the level zero conditions $F^{(A)}(\underline{a}_2) = F^{(A)}(\underline{a}_3) = 0$. The remaining condition in (4.22) is then implied by the other level one super self-duality equations.

4.2 $H = G_2 \subset \mathrm{Spin}(7)$

Using two copies of spinor representation $S = (001)_{\mathbf{8}}$ and the vector $V = (100)_{\mathbf{7}}$, we have the Spin(7) representation spaces

$$
\begin{aligned}
U \in \{S \otimes S\} &= \{U_3 = (002)_{\mathbf{35}}\,,\, U_2 = (010)_{\mathbf{21}}\,,\, U_1 = V = (100)_{\mathbf{7}}\,,\, U_0 = (000)_{\mathbf{1}}\}\,,\\
T \in \{S \otimes V\} &= \{T_1 = (101)_{\mathbf{48}}\,,\; T_2 = (001)_{\mathbf{8}}\}\,,\\
A = \wedge^2 V &= (010)_{\mathbf{21}}\,,
\end{aligned}
\tag{4.24}
$$

which determine the Spin(7) covariant supercurvatures components. Under the breaking $\mathrm{Spin}(7) \supset G_2$ the decompositions of the relevant representation spaces are tabulated below:

X	$\rho_{G_2}(X)$
$U_3 = (002)_{\mathbf{35}}$	$\{\underline{u}_{31} = (02)_{\underline{\mathbf{27}}}\,,\; \underline{u}_{32} = (01)_{\underline{\mathbf{7}}}\,,\; \underline{u}_{33} = (00)_{\underline{\mathbf{1}}}\}$
$U_2 = (010)_{\mathbf{21}}$	$\{\underline{u}_{21} = (10)_{\underline{\mathbf{14}}}\,,\; \underline{u}_{22} = (01)_{\underline{\mathbf{7}}}\}$
$U_1 = V = (100)_{\mathbf{7}}$	$\{\underline{u}_1 = \underline{v} = (01)_{\underline{\mathbf{7}}}\}$
$U_0 = (000)_{\mathbf{1}}$	$\{\underline{u}_0 = (00)_{\underline{\mathbf{1}}}\}$
$T_1 = (101)_{\mathbf{48}}$	$\{\underline{t}_{11} = (02)_{\underline{\mathbf{27}}}\,,\; \underline{t}_{12} = (10)_{\underline{\mathbf{14}}}\,,\; \underline{t}_{13} = (01)_{\underline{\mathbf{7}}}\}$
$T_2 = S = (001)_{\mathbf{8}}$	$\{\underline{t}_{21} = \underline{s}_1 = (01)_{\underline{\mathbf{7}}}\,,\; \underline{t}_{22} = \underline{s}_2 = (00)_{\underline{\mathbf{1}}}\}$
$A = (010)_{\mathbf{21}}$	$\{\underline{a}_1 = (10)_{\underline{\mathbf{14}}}\,,\; \underline{a}_2 = (01)_{\underline{\mathbf{7}}}\}$.

The 35-dimensional completely antisymmetric T_{MNPQ} tensor belongs to the representation $(002)_{\mathbf{35}}$ which contains the G_2 singlet $\underline{u}_{33}$. It can be expressed [1] in terms of the G_2-invariant structure constants C_{MNP} of the algebra of the imaginary octonions as $T_{MNPQ} = \frac{1}{3!}\epsilon_{MNPQRST}C_{RST}$. The curvature $F^{(A)}$ decomposes into two eigenspaces corresponding to eigenvalues $\lambda = 1$ and $\lambda = -2$. In order to investigate the super self-duality conditions, the relevant tensor products of G_2 representations are:

$$
\begin{aligned}
(01)_{\underline{\mathbf{7}}} \otimes (01)_{\underline{\mathbf{7}}} &= (02)_{\underline{\mathbf{27}}} \oplus (10)_{\underline{\mathbf{14}}} \oplus (01)_{\underline{\mathbf{7}}} \oplus (00)_{\underline{\mathbf{1}}}\,,\\
(01)_{\underline{\mathbf{7}}} \otimes (10)_{\underline{\mathbf{14}}} &= (11)_{\underline{\mathbf{64}}} \oplus (02)_{\underline{\mathbf{27}}} \oplus (01)_{\underline{\mathbf{7}}}\,,\\
(01)_{\underline{\mathbf{7}}} \otimes (02)_{\underline{\mathbf{27}}} &= (03)_{\underline{\mathbf{77}}} \oplus (11)_{\underline{\mathbf{64}}} \oplus (02)_{\underline{\mathbf{27}}} \oplus (10)_{\underline{\mathbf{14}}} \oplus (01)_{\underline{\mathbf{7}}}\,.
\end{aligned}
\tag{4.25}
$$

Since V is irreducible, $\overset{\triangledown}{\sigma}_{G_2,\lambda}(V,V) = \{\underline{v}\} = \rho_{G_2}(V)$, and we need to impose $F^{(V)}(\underline{v}) = 0$ for all λ's.

$\boldsymbol{\lambda_{\underline{14}} = 1}$. The eigenspace $\rho_{G_2}(\lambda = 1) = \{\underline{a}_1 = (10)_{\underline{\mathbf{14}}}\}$ corresponds to 7 conditions on the 21 curvatures:

$$
F^{(A)}(\underline{a}_2) = 0\,.
\tag{4.26}
$$

Now, in this case,

$$
\overset{\triangledown}{\sigma}_{G_2,\lambda=1}(S,T_i) = \rho_{G_2}(T_i)\,, \qquad i = 1,2
\tag{4.27}
$$

so $F^{(T)i}(\underline{t}) = 0$ for every T and $\underline{t}$. Imposing the latter would imply that all of $F^{(A)}$ is zero, so there are no algebraic lower level sufficient conditions for (4.26). However, replacing $F^{(T_1)i}(\underline{t}_{12}) = 0$ by the chirality condition

$$\left[D^{(S)i}(\underline{s}_1), F^{(T_1)i}(\underline{t}_{12})\right] = 0 \tag{4.28}$$

yields a non-zero contribution to $F^{(A)}(\underline{a}_1)$ from $[D^{(S)i}(\underline{s}_2), F^{(T_1)i}(\underline{t}_{12})] \neq 0$.

$\boldsymbol{\lambda_{\underline{7}} = -2.}$ In this case we have

$$F^{(A)}(\underline{a}_1) = 0\,, \tag{4.29}$$

which represents 14 conditions on the 21 curvatures, implying $F^{(A)}(\underline{a}_2) \neq 0$. Now,

$$\overset{\triangledown}{\sigma}_{G_2,\lambda=-2}(S,T) = \{\underline{t}_{11}, \underline{t}_{12}, \underline{t}_{13}, \underline{t}_{21}\}\,, \tag{4.30}$$

so the required conditions are,

$$F^{(V)}(\underline{v}) = F^{(T_1)i}(\underline{t}_{11}) = F^{(T_1)i}(\underline{t}_{12}) = F^{(T_1)i}(\underline{t}_{13}) = F^{(T_2)i}(\underline{t}_{21}) = 0\,. \tag{4.31}$$

There remains a single free part, the G_2 singlet $F^{(T_2)}(\underline{t}_{22})$, which contributes to the non-vanishing $F^{(A)}(\underline{a}_2)$. There are no non-trivial level two conditions.

5. Case of $d = 9, 10, 11 \pmod 8$

The dimensions $d = 9, 10, 11 \pmod 8$ with $d \geq 9$ are distinguished by the fact that the vector occurs in the symmetrical square $S \vee S$ of any fundamental spinor representation $S \in \Sigma$. This is actually the simplest case to analyse, since it suffices to consider only one copy of S; the "minimal" case is $N = 1$. Spin(d) for d odd (here $d = 9, 11 \pmod 8$) has only one irreducible fundamental spinor representation S. However, Spin(d) for $d = 10 \pmod 8$ has two irreducible fundamental spinor representations S^+ and S^-, with the vector arising in both $S^+ \vee S^+$ and $S^- \vee S^-$. We will again only consider the *chiral* superspace, in which the S^- representation does not act and we denote S^+ by S.

The curvatures are defined by

$$\begin{aligned}
\left\{D^{(S)}_B, D^{(S)}_C\right\} &= C(S,B,S,C;V,M)\; D^{(V)}_M + \sum_{U \in \{S \vee S\}} C(S,B,S,C;U,L)\; F^{(U)}_L\,,\\
\left[D^{(S)}_B, D^{(V)}_M\right] &= \sum_{T \in \{S \otimes V\}} C(S,B,V,M;T,D)\; F^{(T)}_D\,,\\
\left[D^{(V)}_M, D^{(V)}_N\right] &= F^{(A)}_{MN}\,,
\end{aligned} \tag{5.1}$$

where the Clebsch-Gordon coefficients $C(S,B,S,C;U,L)$ are symmetrical in B, C. From (A.5) and (A.38), the U's are given by

$$\begin{aligned}
&U \in \left\{R(2\pi_r)\,,\ R(\pi_{4p+1})\,;\ 0 \leq p \leq \frac{(r-5)}{4}\right\}, \qquad \text{for } d = 10 \mod 8\,,\\
&U \in \left\{R(2\pi_r)\,,\ R(\pi_{r-4p})\,,\ R(\pi_{r+1-4p})\,;\ 1 \leq p \leq \left[\frac{r}{4}\right]\right\}, \qquad \text{for } d = 9, 11 \mod 8
\end{aligned} \tag{5.2}$$

and the T's are given by (4.3).

The analysis of the two relevant Jacobi identities follows that for the previous cases. The level one identities yield

$$F^{(A)}_{MN} = \frac{1}{2}\left(\left[D^{(V)}_N, F^{(V)}_M\right] - \left[D^{(V)}_M, F^{(V)}_N\right]\right) + \\ + \alpha_1 C(A, MN; S, B, T_1, D)\left\{D^{(S)}_B, F^{(T_1)}_D\right\} + \\ + \alpha_2 C(A, MN; S, B, T_2, C)\left\{D^{(S)}_B, F^{(T_2)}_C\right\}, \tag{5.3}$$

with α_i appropriate recoupling coefficients. This yields the level one constraints:

$$F^{(V)}(\underline{v}) = 0, \qquad \text{for all } \underline{v} \in \overset{\triangledown}{\sigma}_{H,\lambda}(V,V), \tag{5.4}$$

$$F^{(T)}(\underline{t}) = 0, \qquad \text{for all } \underline{t} \in \overset{\triangledown}{\sigma}_{H,\lambda}(S,T). \tag{5.5}$$

In order to have a non-trivial $F^{(A)}$ satisfying (1.1), we require, in addition, that after imposing (5.4) and (5.5) we still have

$$F^{(A)}(\underline{a}) \neq 0, \qquad \text{for at least one } \underline{a} \in \rho_H(\lambda). \tag{5.6}$$

As in the previous case, this is guaranteed if the set of curvature components in (4.8) is non empty.

The second level bare Jacobi identity between $\{D^{(S)}_B, D^{(S)}_C, D^{(S)}_D\}$ reads

$$\sum_{\text{cyclic in } A,B,C}\Bigg(C(S,A,S,B;V,N)\sum_{T\in\{V\otimes S\}} C(S,C,V,N;T,D)\ F^{(T)}_D + \\ + \sum_{U\in\{S\vee S\}} C(S,A,S,B;U,L)\left[D^{(S)}_C, F^{(U)}_L\right]\Bigg) = 0. \tag{5.7}$$

Here the relevant $(S\otimes U)$ tensor products are (A.32) and (A.34) for even d and (A.44) and (A.45) for odd d. These show that $S\otimes U$ contain both T_1 and T_2 except for $R(2\pi_r)\otimes R(\pi_r)$ (for $d = 10 \pmod 8$) which does not yield T_2. The identity (5.7) can be decomposed, after suitable reorganisation, into two pieces:

$$F^{(T_1)}_D = \alpha_3 C(T_1, D; S, B, U_1, M)\left[D^{(S)}_B, F^{(U_1)}_M\right] + \\ + \alpha_4 C(T_1, D; S, B, U_4, P)\left[D^{(S)}_B, F^{(U_4)}_P\right] \tag{5.8}$$

and

$$F^{(T_2)}_C = \alpha_5\left[D^{(S)}_C, F^{(U_0)}\right] + \alpha_6 C(T_2, C; S, B, U_1, M)\left[D^{(S)}_B, F^{(U_1)}_M\right] + \\ + \alpha_7 C(T_2, C; S, B, U_4, P)\left[D^{(S)}_B, F^{(U_4)}_P\right], \tag{5.9}$$

where the α_i are again recoupling coefficients, $U_0 \equiv R(\pi_0)$, the singlet, $U_1 \equiv V$ and $U_4 \equiv \wedge^4 V$ (see appendix A). These yield the level two set of sufficient conditions for the satisfaction of (5.5), namely,

$$F^{(U)}(\underline{u}) = 0, \qquad \text{for all } \underline{u} \in \overset{\triangledown}{\tau}_{H,\lambda}(S,T;S,U), \tag{5.10}$$

with the nontriviality condition that the following set of superfields is non empty:

$$\left\{ F^{(U)}(\underline{u}) \,\middle|\, \underline{u} \in \breve{\tau}_{H,\lambda}(S,T;S,U) \right\}. \tag{5.11}$$

Summarising, we note that either the system of eqs. {(5.4) and (5.5)} or the system {(5.4) and (5.10)} provide sufficient conditions for the satisfaction of the self-duality eqs. (1.1).

5.1 $H = \mathrm{SO}(3) \subset \mathrm{Spin}(9)$

Using the spinor representation $S = (0001)$ and vector $V = (1000)$, we obtain the relevant Spin(9) representation spaces,

$$\begin{aligned} U \in \{S \vee S\} &= \{U_4 = (0002)_{\mathbf{126}}\,, \qquad U_1 = V = (1000)_{\mathbf{9}}\,, \qquad U_0 = (0000)_{\mathbf{1}}\}\,, \\ T \in \{S \otimes V\} &= \{T_1 = (1001)_{\mathbf{128}}\,, \qquad T_2 = S = (0001)_{\mathbf{16}}\}\,, \\ A = V \wedge V &= (0100)_{\mathbf{36}}\,. \end{aligned} \tag{5.12}$$

Under the breaking $\mathrm{Spin}(9) \supset \mathrm{SO}(3)$ the decompositions of these representation spaces are tabulated below. Here we denote SO(3) representations by their dimensions $\underline{\mathbf{d}}$, rather than using their Dynkin indices ($\underline{\mathbf{d}}$-1) or their spins $s = (\underline{\mathbf{d}} - 1)/2$.

X	$\rho_{\mathrm{SO}(3)}(X)$
$U_4 = (0002)_{\mathbf{126}}$	$\{\underline{\mathbf{21}}, \underline{\mathbf{17}}, \underline{\mathbf{15}}, \underline{\mathbf{13}}, \underline{\mathbf{13}}, \underline{\mathbf{11}}, \underline{\mathbf{9}}, \underline{\mathbf{9}}, \underline{\mathbf{7}}, \underline{\mathbf{5}}, \underline{\mathbf{5}}, \underline{\mathbf{1}}\}$
$U_1 = V = (1000)_{\mathbf{9}}$	$\{\underline{\mathbf{9}}\}$
$U_0 = (0000)_{\mathbf{1}}$	$\{\underline{\mathbf{1}}\}$
$T_1 = (1001)_{\mathbf{128}}$	$\{\underline{\mathbf{19}}, \underline{\mathbf{17}}, \underline{\mathbf{15}}, \underline{\mathbf{13}}, \underline{\mathbf{13}}, \underline{\mathbf{11}}, \underline{\mathbf{9}}, \underline{\mathbf{9}}, \underline{\mathbf{7}}, \underline{\mathbf{7}}, \underline{\mathbf{5}}, \underline{\mathbf{3}}\}$
$T_2 = S = (0001)_{\mathbf{16}}$	$\{\underline{\mathbf{11}}, \underline{\mathbf{5}}\}$
$A = (0100)_{\mathbf{36}}$	$\{\underline{\mathbf{15}}, \underline{\mathbf{11}}, \underline{\mathbf{7}}, \underline{\mathbf{3}}\}$

The completely antisymmetric T_{MNPQ} tensor defining self duality belongs to the unique singlet of U_4. The decomposition of the adjoint representation leads to the four eigenvalues $\lambda_{\underline{\mathbf{15}}} = 1$, $\lambda_{\underline{\mathbf{11}}} = -5/8$, $\lambda_{\underline{\mathbf{7}}} = -7/4$ and $\lambda_{\underline{\mathbf{3}}} = 11/8$ (see appendix B). We note that

$$\begin{aligned} \overset{\triangledown}{\sigma}_{\mathrm{SU}(2),\lambda}(V,V) &= \{\underline{v}\} = \rho_{\mathrm{SU}(2)}(V)\,, \\ \overset{\triangledown}{\sigma}_{\mathrm{SU}(2),\lambda}(S,T_i) &= \rho_{\mathrm{SU}(2)}(T_i)\,, \qquad i = 1,2 \end{aligned} \tag{5.13}$$

irrespective of λ. This means that the level one constraints are all trivial. There exist, however, chirality conditions or non-maximal replacements. We describe the latter for all four eigenvalues.

$\boldsymbol{\lambda_{\underline{15}} = 1}$. In order to isolate the $\underline{\mathbf{15}}$, two possibilities present themselves:
a)

$$\begin{aligned} F^{(V)} &= F^{(T_2)} = 0\,, \\ F^{(T_1)}(\underline{t}_1) &= 0\,, \qquad \text{for } \underline{t}_1 \neq \underline{\mathbf{19}}\,, \\ F^{(A)}(\underline{\mathbf{11}}) &= 0 \end{aligned} \tag{5.14}$$

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b)

$$\begin{aligned} F^{(V)} &= F^{(T_2)} = 0\,, \\ F^{(T_1)}(\underline{t}_1) &= 0\,, \qquad \text{for } \underline{t}_1 \neq \underline{\mathbf{17}}\,, \\ F^{(A)}(\underline{a}) &= 0\,, \qquad \text{for } \underline{a} = \{\underline{\mathbf{11}}, \underline{\mathbf{7}}\}\,. \end{aligned} \tag{5.15}$$

$\boldsymbol{\lambda_{\underline{11}} = -5/8}$**.** Here there are three non-maximal replacements:
a)

$$\begin{aligned} F^{(V)} &= F^{(T_2)} = 0\,, \\ F^{(T_1)}(\underline{t}_1) &= 0\,, \qquad \text{for } \underline{t}_1 \neq \underline{\mathbf{3}}\,, \\ F^{(A)}(\underline{a}) &= 0\,, \qquad \text{for } \underline{a} = \{\underline{\mathbf{7}}, \underline{\mathbf{3}}\} \end{aligned} \tag{5.16}$$

b)

$$\begin{aligned} F^{(V)} &= F^{(T_2)} = 0\,, \\ F^{(T_1)}(\underline{t}_1) &= 0\,, \qquad \text{for } \underline{t}_1 \neq \underline{\mathbf{19}}\,, \\ F^{(A)}(\underline{\mathbf{15}}) &= 0 \end{aligned} \tag{5.17}$$

c)

$$\begin{aligned} F^{(V)} &= F^{(T_2)} = 0\,, \\ F^{(T_1)}(\underline{t}_1) &= 0\,, \qquad \text{for } \underline{t}_1 \neq \underline{\mathbf{17}}\,, \\ F^{(A)}(\underline{a}) &= 0\,, \qquad \text{for } \underline{a} = \{\underline{\mathbf{15}}, \underline{\mathbf{7}}\}\,. \end{aligned} \tag{5.18}$$

$\boldsymbol{\lambda_{\underline{7}} = -7/4}$**.** Here the following non-maximal replacements exist:
a)

$$\begin{aligned} F^{(V)} &= F^{(T_2)} = 0\,, \\ F^{(T_1)}(\underline{t}_1) &= 0\,, \qquad \text{for } \underline{t}_1 \neq \underline{\mathbf{3}}\,, \\ F^{(A)}(\underline{a}) &= 0\,, \qquad \text{for } \underline{a} = \{\underline{\mathbf{11}}, \underline{\mathbf{3}}\} \end{aligned} \tag{5.19}$$

b)

$$\begin{aligned} F^{(V)} &= F^{(T_2)} = 0\,, \\ F^{(T_1)}(\underline{t}_1) &= 0\,, \qquad \text{for } \underline{t}_1 \neq \underline{\mathbf{17}}\,, \\ F^{(A)}(\underline{a}) &= 0\,, \qquad \text{for } \underline{a} = \{\underline{\mathbf{15}}, \underline{\mathbf{11}}\}\,. \end{aligned} \tag{5.20}$$

$\boldsymbol{\lambda_{\underline{3}} = 11/8}$**.** For this eigenvalue only one non-maximal replacements of the above type exists:

$$\begin{aligned} F^{(V)} &= F^{(T_2)} = 0\,, \\ F^{(T_1)}(\underline{t}_1) &= 0\,, \qquad \text{for } \underline{t}_1 \neq \underline{\mathbf{3}}\,, \\ F^{(A)}(\underline{a}) &= 0\,, \qquad \text{for } \underline{a} = \{\underline{\mathbf{11}}, \underline{\mathbf{7}}\}\,. \end{aligned} \tag{5.21}$$

6. Concluding remarks

Self-duality equations for Yang-Mills vector potentials in euclidean spaces of dimension d greater than four are first order equations for the vector potential, which take the form of linear constraints on the components of the field strength tensor (1.1) and imply the Yang-Mills equations in virtue of the Jacobi identities. We have investigated possible supersymmetrisations of these self-duality equations. In a manifestly supercovariant d-dimensional euclidean superspace framework, we have developed a scheme for finding systems of sufficient first order equations for the vector and spinor gauge potentials in superspace, which imply, as a consequence of the super Jacobi identities, the self-duality eqs. (1.1) for the vector-vector component of the supercurvature (transforming according to the adjoint representation of Spin(d)). These super self-duality equations are simple linear conditions on the (vector-spinor and spinor-spinor) supercurvature components. In fact, we investigate a chain of implications between three types of superspace equations:

(i) The (level zero) self-duality eqs. (1.1) for the field strength superfields F_{MN} (i.e. vector-vector components of the supercurvature associated to the d superfield vector potentials A_M).

(ii) The level one super self-duality equations imposing linear conditions on certain vector-spinor and spinor-spinor components of the supercurvature (associated to the bosonic vector and fermionic spinor potentials).

(iii) The level two super self-duality equations imposing linear conditions on certain other vector-spinor and spinor-spinor components of the supercurvature.

We know that (i) implies the source-free Yang-Mills equations $D_M F_{MN} = 0$ in virtue of the level zero super Jacobi identities (amongst three vectorial covariant derivatives). We show that (ii) implies (i) as a consequence of level one super Jacobi identities (amongst one vectorial and two spinorial covariant derivatives) and in turn, (iii) implies (ii) as a consequence of level two super Jacobi identities (amongst three spinorial covariant derivatives). Our approach is Lie algebraic, making crucial use of the representation theory of the stability subgroup $H \subset$ Spin(d) of eqs. (1.1). We have discussed some explicit examples for groups of low rank. The familiar N-extended 4-dimensional case has been described at great length, since it is a very precise and simple showcase for our construction.

It remains to see whether our super self-duality equations unambiguously determine the coefficients (depending on the even x coordinates), in an expansion in the odd $(\theta, \bar{\theta})$ variables, of the superfield vector and spinor potentials. Such a component analysis would be necessary in order to investigate the relationship of our systems with supersymmetric BPS conditions, which are defined in terms of component (i.e. x-space) fields, with bosonic fields satisfying self-duality equations like (1.1). A further open question is whether any of our systems of super self-duality afford interpretation as integrability conditions for supersymmetric systems of first order linear equations involving one or more complex parameter.

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A. Some properties of irreducible $\mathrm{Spin}(d)$ representations

In this appendix we collect some useful material about representations of irreducible $\mathrm{Spin}(d)$ representations, partly obtained from [6, 7] and checked using the program *wei.for* written by Jürgen Fuchs [8]. We denote the irreducible representation with highest weight π by $R(\pi)$, the i-th fundamental weight by π_i, $1 \le i \le r$, where the rank of the group $r = [d/2]$, $d = \dim R(\pi_1)$, and $\pi_0 = 0$. Thus, in terms of Dynkin indices $R(\pi_i) = (0\ldots 010 \ldots 0)$, with the 1 at the i-th position. The scalar is $R(\pi_0) = (0 \ldots 0)$ and the vector $V = R(\pi_1) = (10 \ldots 0)$. For all $\mathrm{Spin}(d)$, the symmetric ($\wedge$) and skew symmetric ($\vee$) direct products of V with V are given by,

$$
\begin{aligned}
V \wedge V &= \wedge^2 R(\pi_1) = \begin{cases} R(2\pi_2) = (02)\,, & \text{for } d = 5\ , \\ R(\pi_2+\pi_3) = (011)\,, & \text{for } d = 6\ , \\ R(\pi_2)\,, & \text{for } d \ge 7\,, \end{cases} \\
V \vee V &= \vee^2 R(\pi_1) = R(2\pi_1) \oplus R(\pi_0)\,.
\end{aligned}
\tag{A.1}
$$

A.1 $\mathrm{Spin}(2r)$, $r \ge 3$

For these groups, the p-forms are given by the representations

$$
\wedge^p V = \begin{cases} R(\pi_p)\,, & \text{for } 0 \le p \le r-2\,, \\ R(\pi_{r-1}+\pi_r)\,, & \text{for } p = r-1, r+1\,, \\ R(2\pi_{r-1}) + R(2\pi_r)\,, & \text{for } p = r\,, \\ R(\pi_{2r-p})\,, & \text{for } r+2 \le p \le 2r\,, \end{cases}
\tag{A.2}
$$

where the two irreducible parts of $\wedge^r V$ are the self-dual and anti-self-dual r-forms, $\wedge^r_\pm V$. The tensor products between the vector and spinor representations are given by,

$$
S^+ \otimes V = R(\pi_r) \otimes R(\pi_1) = R(\pi_1+\pi_r) \oplus R(\pi_{r-1}) \equiv T_1^- \oplus T_2^-\,, \tag{A.3}
$$

$$
S^- \otimes V = R(\pi_{r-1}) \otimes R(\pi_1) = R(\pi_1+\pi_{r-1}) \oplus R(\pi_r) \equiv T_1^+ \oplus T_2^+\,. \tag{A.4}
$$

Moreover, the symmetric and skew products of the spinor representations $S^\pm$ are given by,

$$
S^+ \vee S^+ = \vee^2 R(\pi_r) = R(2\pi_r) \bigoplus_{i=1}^{[r/4]} R(\pi_{r-4i}) \equiv U_r^+ \bigoplus_{i=1}^{[r/4]} U^+_{r-4i}\,, \tag{A.5}
$$

$$
S^- \vee S^- = \vee^2 R(\pi_{r-1}) = R(2\pi_{r-1}) \bigoplus_{i=1}^{[r/4]} R(\pi_{r-4i}) \equiv U_r^- \bigoplus_{i=1}^{[r/4]} U^-_{r-4i}\,, \tag{A.6}
$$

$$
S^\pm \wedge S^\pm = \bigoplus_{i=1}^{[r+2/4]} R(\pi_{r+2-4i}) \equiv \bigoplus_{i=1}^{[r+2/4]} U^\pm_{r+2-4i}\,, \tag{A.7}
$$

where $[x]$ denotes the integer part of x. We note that the representations which occur in these decompositions are even forms if the rank is even, $r = 2n$,

$$U_r^{\pm} = \wedge_{\pm}^r V\,, \qquad U_{2p}^+ = U_{2p}^- = \wedge^{2p} V\,, \qquad p = 0, \ldots, n-1\,, \tag{A.8}$$

and odd forms if the rank is odd, $r = 2n+1$,

$$U_r^{\pm} = \wedge_{\pm}^r V\,, \qquad U_{2p+1}^+ = U_{2p+1}^- = \wedge^{2p+1} V\,, \qquad p = 0, \ldots, n-1\,, \tag{A.9}$$

We see that the vector $V = R(\pi_1)$ is contained in $\wedge^2 S^{\pm}$ for $r = 3 \pmod 4$ ($d = 6 \pmod 8$) and in $\vee^2 S^{\pm}$ for $r = 5 \pmod 4 \geq 5$ ($d = 10 \pmod 8$).

A.1.1 Spin$(4n)$, $r = 2n \geq 4$

For these groups, the tensor products between the two irreducible fundamental spinor representations yields odd forms,

$$S^- \otimes S^+ = R(\pi_{r-1}) \otimes R(\pi_r) = R(\pi_{r-1} + \pi_r) \bigoplus_{i=1}^{n-1} R(\pi_{r-1-2i}) \equiv W_{r-1} \bigoplus_{p=0}^{n-2} W_{2p+1}\,. \tag{A.10}$$

We see that for $r = 2n$ the vector $V = R(\pi_1)$ is contained in $S^- \otimes S^+$.

Level 1 products of representations in Spin$(4n)$. The tensor products relevant for the level one identities, (i.e. those concerning $T^{\pm} \otimes S^{\pm}$ and $W \otimes V$), belong to

$$\begin{aligned}
W_{r-1} \otimes V &= R(\pi_{r-1} + \pi_r) \otimes R(\pi_1) \\
&= R(\pi_1 + \pi_{r-1} + \pi_r) \oplus R(2\pi_{r-1}) \oplus R(2\pi_r) \oplus R(\pi_{r-2})\,, &\text{(A.11)}\\
W_{2p+1} \otimes V &= R(\pi_{2p+1}) \otimes R(\pi_1)\,, \qquad \text{for } p = 0, \ldots, n-2 \\
&= R(\pi_1 + \pi_{2p+1}) \oplus R(\pi_{2p}) \oplus R(\pi_{2p+2})\,, &\text{(A.12)}\\
T_1^+ \otimes S^+ &= R(\pi_1 + \pi_{r-1}) \otimes R(\pi_r) \\
&= R(\pi_1 + \pi_{r-1} + \pi_r) \oplus R(2\pi_{r-1}) \bigoplus_{i=1}^{n-1} R(\pi_{r-2i}) \bigoplus_{i=1}^{n-1} R(\pi_1 + \pi_{r-1-2i})\,, &\text{(A.13)}\\
T_1^- \otimes S^- &= R(\pi_1 + \pi_r) \otimes R(\pi_{r-1}) \\
&= R(\pi_1 + \pi_{r-1} + \pi_r) \oplus R(2\pi_r) \bigoplus_{i=1}^{n-1} R(\pi_{r-2i}) \bigoplus_{i=1}^{n-1} R(\pi_1 + \pi_{r-1-2i})\,. &\text{(A.14)}
\end{aligned}$$

We see that the adjoint representation $R(\pi_2)$ is contained on the right-hand sides of (A.13), (A.14) and of (A.12) for $p = 0, 1$.

Level 2 products of representations in Spin$(4n)$. The level 2 Jacobi identities tensor products with S^+ are:

$$\begin{aligned}
W_{r-1} \otimes S^+ &= R(\pi_{r-1} + \pi_r) \otimes R(\pi_r) \\
&= R(\pi_{r-1} + 2\pi_r) \bigoplus_{i=1}^{n-1} R(\pi_{r-1-2i} + \pi_r) \bigoplus_{i=1}^{n} R(\pi_{r-2i} + \pi_{r-1})\,, &\text{(A.15)}
\end{aligned}$$

$$
\begin{aligned}
W_{2p+1} \otimes S^+ &= R(\pi_{2p+1}) \otimes R(\pi_r)\,, \qquad \text{for } p = 0, \ldots, n-2 \\
&= \bigoplus_{i=0}^{p} R(\pi_{2p+1-2i} + \pi_r) \bigoplus_{i=0}^{p} R(\pi_{2p-2i} + \pi_{r-1})\,, \qquad \text{(A.16)} \\
U_r^- \otimes S^+ &= R(2\pi_{r-1}) \otimes R(\pi_r) \\
&= R(2\pi_{r-1} + \pi_r) \bigoplus_{i=1}^{n-1} R(\pi_{r-1-2i} + \pi_{r-1})\,, \qquad \text{(A.17)} \\
U_{2p}^- \otimes S^+ &= R(\pi_{2p}) \otimes R(\pi_r)\,, \qquad \text{for } p = 1, \ldots, n-1 \\
&= \bigoplus_{i=0}^{p} R(\pi_{2p-2i} + \pi_r) \bigoplus_{i=0}^{p-1} R(\pi_{2p-1-2i} + \pi_{r-1}) \qquad \text{(A.18)}
\end{aligned}
$$

and products involving S^- are:

$$
\begin{aligned}
W_{r-1} \otimes S^- &= R(\pi_{r-1} + \pi_r) \otimes R(\pi_{r-1}) \\
&= R(2\pi_{r-1} + \pi_r) \bigoplus_{i=1}^{n} R(\pi_{r-2i} + \pi_r) \bigoplus_{i=1}^{n-1} R(\pi_{r-1-2i} + \pi_{r-1})\,, \qquad \text{(A.19)} \\
W_{2p+1} \otimes S^- &= R(\pi_{2p+1}) \otimes R(\pi_{r-1})\,, \qquad \text{for } p = 0, \ldots, n-2 \\
&= \bigoplus_{i=0}^{p} R(\pi_{2p-2i} + \pi_r) \bigoplus_{i=0}^{p} R(\pi_{2p+1-2i} + \pi_{r-1})\,, \qquad \text{(A.20)} \\
U_r^+ \otimes S^- &= R(2\pi_r) \otimes R(\pi_{r-1}) \\
&= R(\pi_{r-1} + 2\pi_r) \bigoplus_{i=1}^{n-1} R(\pi_{r-1-2i} + \pi_r)\,, \qquad \text{(A.21)} \\
U_{2p}^+ \otimes S^- &= R(\pi_{2p}) \otimes R(\pi_{r-1})\,, \qquad \text{for } p = 1, \ldots, n-1 \\
&= \bigoplus_{i=0}^{p-1} R(\pi_{2p-1-2i} + \pi_r) \bigoplus_{i=0}^{p} R(\pi_{2p-2i} + \pi_{r-1})\,. \qquad \text{(A.22)}
\end{aligned}
$$

A.1.2 Spin$(4n+2)$, $r = 2n+1 \geq 3$

In these cases, we really only need, for the examples we have treated corresponding to the chiral case, to use $S = S^+$ and $\{T\} = \{T^-\}$. For completeness, we also give the direct products required to extend our results to non-chiral situations. For these groups, the tensor products between the two irreducible fundamental spinor representations yields even forms,

$$
S^- \otimes S^+ = R(\pi_{r-1}) \otimes R(\pi_r) = R(\pi_{r-1} + \pi_r) \bigoplus_{i=1}^{n} R(\pi_{r-1-2i}) \equiv W_{r-1} \bigoplus_{p=0}^{n-1} W_{2p}\,. \qquad \text{(A.23)}
$$

Level 1 products of representations in Spin$(4n+2)$. The tensor products relevant for the level one identities (concerning $T^\mp \otimes S^\pm$ and $U^\pm \otimes V$) are,

$$
\begin{aligned}
U_r^+ \otimes V &= R(2\pi_r) \otimes R(\pi_1) \\
&= R(\pi_1 + 2\pi_r) \oplus R(\pi_{r-1} + \pi_r)\,, \qquad \text{(A.24)}
\end{aligned}
$$

$$\begin{aligned}
U_r^- \otimes V &= R(2\pi_{r-1}) \otimes R(\pi_1) \\
&= R(\pi_1 + 2\pi_{r-1}) \oplus R(\pi_{r-1} + \pi_r)\,, \qquad (A.25) \\
U_{2p+1}^{\pm} \otimes V &= R(\pi_{2p+1}) \otimes R(\pi_1)\,, \qquad \text{for } p = 0, \ldots, n-2 \\
&= R(\pi_1 + \pi_{2p+1}) \oplus R(\pi_{2p}) \oplus R(\pi_{2p+2})\,, \qquad (A.26) \\
U_{r-2}^{\pm} \otimes V &= R(\pi_{r-2}) \otimes R(\pi_1) \\
&= R(\pi_1 + \pi_{r-2}) \oplus R(\pi_{r-3}) \oplus R(\pi_{r-1} + \pi_r)\,, \qquad (A.27) \\
T_1^- \otimes S^+ &= R(\pi_1 + \pi_r) \otimes R(\pi_r) \\
&= R(\pi_1 + 2\pi_r) \oplus R(\pi_{r-1} + \pi_r) \bigoplus_{i=1}^{n-1} R(\pi_{r-1-2i}) \bigoplus_{i=1}^{n} R(\pi_1 + \pi_{r-2i})\,, \qquad (A.28) \\
T_1^+ \otimes S^- &= R(\pi_1 + \pi_{r-1}) \otimes R(\pi_{r-1}) \\
&= R(\pi_1 + 2\pi_{r-1}) \oplus R(\pi_{r-1} + \pi_r) \bigoplus_{i=1}^{n-1} R(\pi_{r-1-2i}) \bigoplus_{i=1}^{n} R(\pi_1 + \pi_{r-2i})\,. \qquad (A.29)
\end{aligned}$$

We see that for $n > 1$ ($r > 3$) the adjoint representation $R(\pi_2)$ is contained on the right-hand sides of (A.28), (A.29) and of (A.26) for $p = 0, 1$. Analogously, for Spin(6)($r = 3$), the adjoint representation $R(\pi_2 + \pi_3)$ is contained in (A.24), (A.27) and (A.28).

Level 2 products of representations in Spin($4n + 2$). The level 2 Jacobi identities for $d = 2r = 4n + 2$ involve the following tensor products with S^+,

$$\begin{aligned}
W_{r-1} \otimes S^+ &= R(\pi_{r-1} + \pi_r) \otimes R(\pi_r) \\
&= R(\pi_{r-1} + 2\pi_r) \bigoplus_{i=1}^{n} R(\pi_{r-1-2i} + \pi_r) \bigoplus_{i=1}^{n} R(\pi_{r-2i} + \pi_{r-1})\,, \qquad (A.30) \\
W_{2p} \otimes S^+ &= R(\pi_{2p}) \otimes R(\pi_r)\,, \qquad \text{for } p = 1, \ldots, n-1 \\
&= \bigoplus_{i=0}^{p-1} R(\pi_{2p-1-2i} + \pi_{r-1}) \bigoplus_{i=0}^{p} R(\pi_{2p-2i} + \pi_r)\,, \qquad (A.31) \\
U_r^+ \otimes S^+ &= R(2\pi_r) \otimes R(\pi_r) \\
&= R(3\pi_r) \bigoplus_{i=1}^{n} R(\pi_{r-2i} + \pi_r)\,, \qquad (A.32) \\
U_r^- \otimes S^+ &= R(2\pi_{r-1}) \otimes R(\pi_r) \\
&= R(2\pi_{r-1} + \pi_r) \bigoplus_{i=1}^{n} R(\pi_{r-1-2i} + \pi_{r-1})\,, \qquad (A.33) \\
U_{2p+1}^+ \otimes S^+ &= R(\pi_{2p+1}) \otimes R(\pi_r)\,, \qquad \text{for } p = 1, \ldots, n-1 \\
&= \bigoplus_{i=0}^{p} R(\pi_{2p+1-2i} + \pi_r) \bigoplus_{i=0}^{p} R(\pi_{2p-2i} + \pi_{r-1})\,, \qquad (A.34)
\end{aligned}$$

and the tensor products with S^- are,

$$
\begin{aligned}
W_{r-1}\otimes S^- &= R(\pi_{r-1}+\pi_r)\otimes R(\pi_{r-1})\\
&= R(2\pi_{r-1}+\pi_r)\bigoplus_{i=1}^{n}R(\pi_{r-2i}+\pi_r)\bigoplus_{i=1}^{n}R(\pi_{r-1-2i}+\pi_{r-1})\,,\\
W_{2p}\otimes S^- &= R(\pi_{2p})\otimes R(\pi_{r-1})\,,\qquad \text{for } p=1,\ldots,n-1\\
&= \bigoplus_{i=0}^{p-1}R(\pi_{2p-1-2i}+\pi_r)\bigoplus_{i=0}^{p}R(\pi_{2p-2i}+\pi_{r-1})\,,\\
U_r^-\otimes S^- &= R(2\pi_{r-1})\otimes R(\pi_{r-1})\\
&= R(3\pi_{r-1})\bigoplus_{i=1}^{n}R(\pi_{r-2i}+\pi_{r-1})\,,\\
U_r^+\otimes S^- &= R(2\pi_r)\otimes R(\pi_{r-1})\\
&= R(\pi_{r-1}+2\pi_r)\bigoplus_{i=1}^{n}R(\pi_{r-1-2i}+\pi_r)\,,\\
U_{2p+1}^-\otimes S^- &= R(\pi_{2p+1})\otimes R(\pi_{r-1})\,,\qquad \text{for } p=1,\ldots,n-1\\
&= \bigoplus_{i=0}^{p}R(\pi_{2p+1-2i}+\pi_{r-1})\bigoplus_{i=0}^{p}R(\pi_{2p-2i}+\pi_r)\,.
\end{aligned}\tag{A.35}
$$

A.2 Spin$(2r+1), r\geq 2$

For these groups p-forms are given by the representations

$$
\wedge^p V = \begin{cases} R(\pi_p)\,, & \text{for } 0\leq p\leq r-1\,,\\ R(2\pi_r)\,, & \text{for } p=r,r+1\,,\\ R(\pi_{2r+1-p})\,, & \text{for } r+2\leq p\leq 2r+1\,.\end{cases}\tag{A.36}
$$

There is only one S of dimension 2^r and the product of representations appearing in the definitions of the curvatures (fields) are

$$
S\otimes V = R(\pi_r)\otimes R(\pi_1) = R(\pi_1+\pi_r)\oplus R(\pi_r)\equiv T_1\oplus T_2\,,\tag{A.37}
$$

$$
\begin{aligned}
\vee^2 S &= \vee^2 R(\pi_r)\\
&= R(2\pi_r)\bigoplus_{i=1}^{[r/4]}R(\pi_{r-4i})\bigoplus_{i=1}^{[r+1/4]}R(\pi_{r+1-4i})\equiv U_r\bigoplus_{i=1}^{[r/4]}U_{r-4i}\bigoplus_{i=1}^{[r+1/4]}U_{r+1-4i}\,,
\end{aligned}\tag{A.38}
$$

$$
\begin{aligned}
\wedge^2 S &= \wedge^2 R(\pi_r)\\
&= \bigoplus_{i=1}^{[r+2/4]}R(\pi_{r+2-4i})\bigoplus_{i=1}^{[r+3/4]}R(\pi_{r+3-4i})\equiv\bigoplus_{i=1}^{[r+2/4]}U_{r+2-4i}\bigoplus_{i=1}^{[r+3/4]}U_{r+3-4i}\,.
\end{aligned}\tag{A.39}
$$

We note that the vector V the adjoint A and three-form U_3 are contained as follows in $S\otimes S$ (for $p=0,1,\ldots$):

- $V=R(\pi_1)\subset\wedge^2 S$ for $r=2+4p, 3+4p$ $(d=5+8p, 7+8p)$

- $V = R(\pi_1) \subset \vee^2 S$ for $r = 4+4p, 5+4p$ ($d = 9+8p, 11+8p$)
- $A = R(\pi_2) \subset \wedge^2 S$ for $r = 3+4p, 4+4p$ ($d = 7+8p, 9+8p$)
- $A = \wedge^2 V \subset \vee^2 S$ for $r = 2+4p, 5+4p$ ($d = 5+8p, 11+8p$)
- $U_3 = R(\pi_3) \subset \wedge^2 S$ for $r = 4+4p, 5+4p$ ($d = 9+8p, 11+8p$)
- $U_3 = \wedge^3 V \subset \vee^2 S$ for $r = 3+4p, 6+4p$ ($d = 7+8p, 13+8p$) .

Level 1 products of representations in Spin($2r+1$). For these groups, the tensor product relevant for the level one identities involving $T_1 \otimes S$ or $U \otimes V$ are (note that $T_2 = S$ which has already been taken care off),

$$\begin{aligned} T_1 \otimes S &= R(\pi_1 + \pi_r) \otimes R(\pi_r) \\ &= R(\pi_1 + 2\pi_r) \oplus R(2\pi_r) \bigoplus_{i=1}^{r-1} R(\pi_{r-i}) \bigoplus_{i=1}^{r-1} R(\pi_1 + \pi_{r-i}) \,, \end{aligned} \tag{A.40}$$

$$\begin{aligned} U_r \otimes V &= R(2\pi_r) \otimes R(\pi_1) \\ &= R(\pi_1 + 2\pi_r) \oplus R(\pi_{r-1}) \oplus R(2\pi_r) \,, \end{aligned} \tag{A.41}$$

$$\begin{aligned} U_p \otimes V &= R(\pi_p) \otimes R(\pi_1) \,, \qquad \text{for } p = 1, \ldots, r-2 \\ &= R(\pi_1 + \pi_p) \oplus R(\pi_{p-1}) \oplus R(\pi_{p+1}) \,, \end{aligned} \tag{A.42}$$

$$\begin{aligned} U_{r-1} \otimes V &= R(\pi_{r-1}) \otimes R(\pi_1) \\ &= R(\pi_1 + \pi_{r-1}) \oplus R(\pi_{r-2}) \oplus R(2\pi_r) \,. \end{aligned} \tag{A.43}$$

We see that the adjoint $A = R(\pi_2)$ appears in the tensor products: $T_1 \otimes S$, $U_1 \otimes V = V \otimes V$ and $U_3 \otimes V$.

Level 2 products of representations in Spin($2r+1$). The level 2 identities involve, as $U \otimes S$

$$\begin{aligned} U_r \otimes S &= R(2\pi_r) \otimes R(\pi_r) \\ &= R(3\pi_r) \bigoplus_{i=1}^{r} R(\pi_{r-i} + \pi_r) \,, \end{aligned} \tag{A.44}$$

$$\begin{aligned} U_p \otimes S &= R(\pi_p) \otimes R(\pi_r) \\ &= \bigoplus_{i=0}^{p} R(\pi_{p-i} + \pi_r) \,, \qquad \text{for } p = 1, \ldots, r-1 \,. \end{aligned} \tag{A.45}$$

The representations T_1 and $T_2 = S$ appear in all these products.

A.3 Extended Poincaré algebras

Super extensions. In eq. (1.9), the vectorial translation operator is realised as the anticommutator of two spinorial translation operators (*super* extensions). From the tensor products given above, we see that:

S1. For $d = 4p \geq 4$ the vector occurs in the direct product of the two inequivalent spinors S^+, S^-. Hence the minimal model has $N = 1$ and both S^+ and S^- are present.

S2. For $d = 6 + 8p \geq 6$ the vector occurs in $\wedge^2 S^+$ and in $\wedge^2 S^-$. Hence the minimal model has $N = 2$. There exist both chiral possibilities (with two S^+'s or equivalently two S^-) and non-chiral possibilities (with two S^+'s as well as two S^-).

S3. For $d = 10 + 8p \geq 10$ the vector occurs in $\vee^2 S^+$ and in $\vee^2 S^-$. Hence the minimal model has $N = 1$. There exist both chiral and non-chiral possibilities.

S4. For $d = 5, 7 \pmod 8$ the vector occurs in $\wedge^2 S$. Hence the minimal model has $N = 2$.

S5. For $d = 9, 11 \pmod 8$ the vector occurs in $\vee^2 S$. Hence the minimal model has $N = 1$.

Lie extensions. One could also consider *even* extensions of the Poincaré algebra [9], i.e. Z_2-graded *Lie* (rather than *super*) algebras realised on hyperspaces parametrised by entirely even (vectorial and spinoral) coordinates. The vectorial translation generators in such algebras are then obtained from the *commutator* of two spinorial derivatives

$$\left[\nabla_A^{(S_1)}, \nabla_B^{(S_1)}\right] = \left(\Gamma^M\right)_{AB} \nabla_M^{(V)} . \tag{A.46}$$

In such "changed-statistics" cases, the roles of symmetry and skewsymmetry are interchanged. This leads in an obvious fashion to the following pattern:

L1. For $d = 4p \geq 4$ the minimal model has $N = 1$ and both S^+ and S^- are present.

L2. For $d = 6 + 8p \geq 6$ the minimal model has $N = 1$. There exist both chiral and non-chiral possibilities.

L3. For $d = 10 + 8p \geq 10$ the minimal model has $N = 2$. There exist both chiral and non-chiral possibilities.

L4. For $d = 5, 7 \pmod 8$ the minimal model has $N = 1$.

L5. For $d = 9, 11 \pmod 8$ the minimal model has $N = 2$.

Our considerations extend in an obvious fashion to such "changed-statistics" hyperspaces.

B. SO(4)-invariant 4-forms in d-dimensions

Consider an orthogonal group SO(d) of dimension $d = (p + 1)(q + 1)$ ($p + q$ even) and its sugroup SU(2) $\otimes$ SU(2), such that the d-dimensional vector repressentation of SO(d) decomposes into the irreducible (p,q) representation, conventionally called the spin($p/2, q/2$) representation. Choose a basis of weights for the vector representation

$$\begin{aligned} &\left\{A^M ;\, M = 1, \ldots, d\right\} \\ \Longleftrightarrow\; &\left\{A(s, \dot{s}) ;\; s = -\frac{p}{2}, -\frac{p}{2} + 1, \ldots, \frac{p}{2},\; \dot{s} = -\frac{q}{2}, -\frac{q}{2} + 1, \ldots, \frac{q}{2}\right\} , \end{aligned} \tag{B.1}$$

where the correspondence of the indices is given by

$$M = \left(\dot{s} + \frac{q}{2} + 1\right) + (q+1)\left(s + \frac{p}{2}\right). \tag{B.2}$$

Here $s, \dot{s}$ are the eigenvalues of the generators $L_0, \dot{L}_0$ of the Cartan subalgebra,

$$L_0 A(s,\dot{s}) = sA(s,\dot{s}), \qquad \dot{L}_0 A(s,\dot{s}) = \dot{s}A(s,\dot{s}), \tag{B.3}$$

and the action of the weight-raising operators $L_+, \dot{L}_+$ (of the two simple SU(2) factors) is

$$L_+ A(s,\dot{s}) = t(p,s)A(s+1,\dot{s}), \qquad \dot{L}_+ A(s,\dot{s}) = t(q,\dot{s})A(s,\dot{s}+1), \tag{B.4}$$

where $t(p,s) \equiv \sqrt{(p/2+s+1)(p/2-s)}$. In this basis, the SO(4)-invariant scalar product on this representation space is given by

$$\langle A, A\rangle \equiv G_{MN}A^M A^N, \qquad G_{MN} = (-1)^{M+1}\delta(M+N-(d+1)). \tag{B.5}$$

Using the correspondance (B.1), it is easy to check the SO(4)-invariance, i.e. $L_0\langle A,A\rangle = L_\pm\langle A,A\rangle = \dot{L}_0\langle A,A\rangle = \dot{L}_\pm\langle A,A\rangle = 0$.

Now consider a scalar constructed from the real skewsymmetric product of four vectors: $X := T_{MNPQ}A^M \wedge B^N \wedge C^P \wedge D^Q$. Requiring $L_0 X = 0$ and $\dot{L}_0 X = 0$ yields a set of a priori non-zero components of the tensor T_{MNPQ}. These components are determined by the further linear algebraic equations obtained from the coefficients of $A^M B^N C^P D^Q$ in $L_+ X = 0$ and $\dot{L}_+ X = 0$. The relations $L_- X = 0$ and $\dot{L}_- X = 0$ are then automatically satisfied. The number of independent parameters in the thus constructed tensor T_{MNPQ} equals the number of singlets in the decomposition of the $\binom{d}{4}$-dimensional representation of SO(d) into irreducible SO(4) representations. If there are several singlets, then by appropriate choice of the parameters in T_{MNPQ}, the independent invariants may be extracted. To obtain the eigenvalues, we define the symmetric $\binom{d}{2} \times \binom{d}{2}$ matrix V by the correspondence

$$\left\{V_L^K\,;\; K,L = 1,\ldots,\binom{d}{2}\right\}$$
$$\Updownarrow$$
$$\{T_{MN}^{RS} = G^{PR}G^{QS}T_{MNPQ}\,;\; M,N,P,Q,R,S = 1,\ldots,d,\; M<N, R<S\}, \tag{B.6}$$

where the indices labeling the adjoint representation $K, L = 1,\ldots,d(d-1)/2$ are related to the vector indices $R, S, M, N = 1,\ldots,d$ by

$$L = M + \frac{-N^2 + (2d-1)N - 2d}{2}, \qquad K = R + \frac{-S^2+(2d-1)S-2d}{2}. \tag{B.7}$$

Further, the correspondence

$$\left\{F_K\,;\; K = 1,\ldots,\binom{d}{2}\right\} \Longleftrightarrow \{F_{MN}\,;\; M<N = 1,\ldots,d\}, \tag{B.8}$$

allows us to write (1.1), which in the basis (B.1) takes the form

$$\frac{1}{2}G^{PR}G^{QS}T_{MNPQ}F_{RS} = \lambda F_{MN}, \tag{B.9}$$

as an eigenvalue equation

$$V_L^K F_K = \lambda F_L. \tag{B.10}$$

The matrix V may readily be diagonalised to yield the eigenvalues.

Examples.

d=8, $(\mathsf{p},\mathsf{q}) = (1,3)$

The non-zero components of the T-tensor are

$$1 = T_{1278} = T_{3456} = -T_{2457} = \frac{1}{3}T_{2367} = -\frac{1}{2}T_{2358} = -\frac{1}{2}T_{1467} = \frac{1}{3}T_{1458} = -T_{1368}\,.$$

This has three eigenspaces with eigenvalues $\lambda_{\underline{\mathbf{15}}} = 1$, $\lambda_{\underline{\mathbf{10}}} = -3$ and $\lambda_{\underline{\mathbf{3}}} = 5$ corresponding, respectively, to the eigenrepresentations

$$(2,4)_{\underline{\mathbf{15}}}\,, \qquad (0,6)_{\underline{\mathbf{7}}} \oplus (0,2)_{\underline{\mathbf{3}}}\,, \qquad \text{and} \qquad (2,0)_{\underline{\mathbf{3}}}\,.$$

d=9, $(\mathsf{p},\mathsf{q}) = (2,2)$

The non-zero components of the T-tensor are

$$1 = T_{1289} = -T_{1379} = -T_{1469} = T_{2459} = T_{2378} = -T_{3458} = T_{1568} = -T_{2567} = T_{3467}\,.$$

This completely splits the space of bi-vectors into its irreducible parts:

$$\lambda_{(2,4)} = 1\,, \qquad \lambda_{(4,2)} = -1\,, \qquad \lambda_{(0,2)} = 2\,, \qquad \text{and} \qquad \lambda_{(2,0)} = -2\,.$$

d=7, $(\mathsf{p},\mathsf{q}) = (6,0)$

The tensor with non-zero components:

$$1 = T_{1267} = -T_{1357} = -T_{2356} = \frac{1}{\sqrt{2}}T_{1456} = \frac{1}{\sqrt{2}}T_{2347}$$

has eigenvalues $\lambda_{(10)\oplus(2)} = 1$ and $\lambda_{(6)} = -2$.

d=9, $(\mathsf{p},\mathsf{q}) = (8,0)$

The tensor with non-zero components:

$$\begin{aligned}1 &= T_{1289} = -T_{1379} = \frac{4}{3}T_{1478} = -\frac{2\sqrt{2}}{\sqrt{5}}T_{1568} = 2T_{1469} = -8T_{2378} = -\frac{8}{3}T_{2468}\\ &= -\frac{2\sqrt{2}}{\sqrt{5}}T_{2459} = \frac{8}{\sqrt{70}}T_{2567} = \frac{8}{\sqrt{70}}T_{3458} = -\frac{8}{7}T_{3467} = \frac{4}{3}T_{2369}\end{aligned}$$

has irreducible eigenrepresentations with eigenvalues:

$$\lambda_{(14)} = 1\,, \qquad \lambda_{(10)} = -\frac{5}{8}\,, \qquad \lambda_{(6)} = -\frac{7}{4}\,, \qquad \text{and} \qquad \lambda_{(2)} = \frac{11}{8}\,.$$

d=12, (p,q)= (1,5)

Here there are two invariant T-tensors (we now use commas to separate the indices):

$$
\begin{aligned}
1 &= T_{1,2,11,12} = -T_{1,3,10,12} = T_{1,4,9,12} = -T_{1,5,8,12} = \frac{1}{3}T_{1,6,7,12} = -\frac{1}{2}T_{1,6,8,11} \\
&= \frac{1}{2}T_{1,6,9,10} = T_{2,3,10,11} = -T_{2,4,9,11} = -\frac{1}{2}T_{2,5,7,12} = \frac{1}{3}T_{2,5,8,11} \\
&= -\frac{1}{2}T_{2,5,9,10} = -T_{2,6,7,11} = \frac{1}{2}T_{3,4,7,12} = -\frac{1}{2}T_{3,4,8,11} = \frac{1}{3}T_{3,4,9,10} \\
&= -T_{3,5,8,10} = T_{3,6,7,10} = T_{4,5,8,9} = -T_{4,6,7,9} = T_{5,6,7,8}\,, \\
1 &= \frac{1}{5}T_{1,4,9,12} = -\frac{1}{3\sqrt{5}}T_{1,4,10,11} = -\frac{1}{10}T_{1,5,8,12} = \frac{1}{2\sqrt{10}}T_{1,5,9,11} = \frac{1}{20}T_{1,6,7,12} \\
&= -\frac{1}{10}T_{1,6,8,11} = \frac{1}{5}T_{1,6,9,10} = -\frac{1}{3\sqrt{5}}T_{2,3,9,12} = \frac{1}{9}T_{2,3,10,11} = \frac{1}{2\sqrt{10}}T_{2,4,8,12} \\
&= \frac{1}{4}T_{2,4,9,11} = -\frac{1}{10}T_{2,5,7,12} = \frac{1}{12}T_{2,5,8,11} = -\frac{1}{13}T_{2,5,9,10} = -\frac{1}{10}T_{2,6,7,11} \\
&= \frac{1}{2\sqrt{10}}T_{2,6,8,10} = \frac{1}{5}T_{3,4,7,12} = -\frac{1}{13}T_{3,4,8,11} = \frac{1}{17}T_{3,4,9,10} = \frac{1}{2\sqrt{10}}T_{3,5,7,11} \\
&= -\frac{1}{4}T_{3,5,8,10} = \frac{1}{5}T_{3,6,7,10} = -\frac{1}{3\sqrt{5}}T_{3,6,8,9} = -\frac{1}{3\sqrt{5}}T_{4,5,7,10} = \frac{1}{9}T_{4,5,8,9}\,.
\end{aligned}
\tag{B.11}
$$

The corresponding eigenvalues for the irreducible representation spaces (denoted here by dimension) are found to be:

$$
\begin{array}{ccccccccccccc}
66 & = & \underline{\mathbf{27}} & + & \underline{\mathbf{15}} & + & \underline{\mathbf{11}} & + & \underline{\mathbf{7}} & + & \underline{\mathbf{3}} & + & \underline{\mathbf{3}}\,, \\
\lambda_1 & : & 1 & & 1 & & -3 & & -3 & & -3 & & 7\,, \\
\lambda_2 & : & 0 & & 14 & & -20 & & -2 & & -27 & & 35\,.
\end{array}
\tag{B.12}
$$

d=15, (p,q)= (2,4)

Here there are again two invariant tensors, with eigenvalues:

$$
\begin{array}{ccccccccccccccc}
105 & = & \underline{\mathbf{35}} & + & \underline{\mathbf{27}} & + & \underline{\mathbf{15}} & + & \underline{\mathbf{15}} & + & \underline{\mathbf{7}} & + & \underline{\mathbf{3}} & + & \underline{\mathbf{3}}\,, \\
\lambda_1 & : & 1 & & -1 & & 1 & & -1 & & -2 & & -2 & & 4\,, \\
\lambda_2 & : & 0 & & 4 & & -5 & & -3 & & 0 & & 10 & & -6\,.
\end{array}
\tag{B.13}
$$

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